The Academic Algebra

Bradbury — Emery

Eaton and Bradbury's Mathematical Series.

THE

ACADEMIC ALGEBRA.

BY

WILLIAM F. BRADBURY, A. M.,
HEAD MASTER OF THE CAMBRIDGE LATIN SCHOOL,

AND

GRENVILLE C. EMERY, A. M.,
MASTER IN THE BOSTON LATIN SCHOOL.

BOSTON:
THOMPSON, BROWN, AND COMPANY,
23 HAWLEY STREET.

University Press:
JOHN WILSON AND SON, CAMBRIDGE.

PREFACE.

THE favor with which the Eaton and Bradbury's Mathematical Series has been received has encouraged the authors to prepare this book. It is designed to meet the demand for a fuller treatment of Factoring, for more numerous examples for practice, and for the more advanced work now required in our best High Schools and Academies, and for admission to many of the Colleges.

In all the subjects treated the exercises have been carefully graded to lead the student from the simple to the more difficult.

Special attention is invited to the treatment of Positive and Negative Numbers in Chapter II., of Addition and Subtraction as a single topic, as well as Multiplication and Division, in both Integral and Fractional Numbers; to the arrangement of the equations in Elimination; to the interpretation of negative results, and of the forms, $\frac{0}{A}$, $\frac{A}{0}$, and $\frac{0}{0}$; and to the treatment of Affected Quadratic Equations.

Although a work on Mathematics has a natural order for the development of topics, yet this may not be the best order for the pupil. To awaken the pupil's interest in algebraic operations, a few problems have been introduced in Chapter I., and to keep this interest alive, and

give him some idea of the beauty and utility of Algebra in its application, teachers are recommended, after completing Chapters I. and II., to pass over to Chapter XI., and while going over this and Chapter XII. to take the part omitted. It may also be better for the younger pupils to omit the demonstration in Art. 108, and the Theorems in Arts. 121 and 122, until they become more familiar with algebraic reasoning.

The adjective, Arithmétic, is used by recent writers on Algebra, and in referring to a *series* of numbers, as it corresponds with the adjectives, Geometric and Harmonic, it has been adopted in this work.

At the end of the book are the Examination Questions in Algebra for admission to several New England Colleges, from September, 1884, to September, 1888, inclusive.

W. F. B.
G. C. E.

CAMBRIDGE, MASS., June, 1889.

NOTE. Great care has been taken to avoid errors. If any are found, a statement of them to the authors will be thankfully received.

CONTENTS.

ALGEBRA.

CHAPTER I.

DEFINITIONS AND NOTATION.

1. **Algebra** is the science which treats of numbers.

In Arithmetic the numbers are positive, and represented by figures; while in Algebra numbers may be positive or negative, real or imaginary, and represented by figures or letters.

In most algebraic operations these numbers have an abstract signification; that is, they represent the *measure*, absolute or approximate, of some quantity referred to a unit of its own kind, arbitrarily selected as the standard.

Through the annexation of the *name* of the measuring unit they become concrete.

NOTE 1. **Quantity**, the subject of all mathematical investigation, has been variously used to mean not only anything which can be measured, — as time, distance, etc., — but also the abstract number arising from its measurement, and the concrete number representing its measurement, together with their various combinations. "It is any symbol which results from the rules of calculation." (De Morgan.)

In this work the word "quantity" will be used to mean the thing measured, or the concrete number representing its measurement. When the abstract number arising from its measurement is clearly meant, the word "number" will be used.

2. The first letters of the alphabet, a, b, c, etc., generally stand for what are called *known* numbers, — that is, those whose values are *given;* and the last letters, x, y, z, etc., for *unknown* numbers, — that is, those whose values are *to be determined.* Accented and subscript letters, as r', r'', etc. (read, *r prime, r second,* etc.), or r_1, r_2, etc. (read, *r sub one, r sub two,* etc.), are often used to represent numbers of the same kind, but of different values; as the terms of a continued proportion, or different rates of interest.

Each letter may represent any number whatever, but throughout the same investigation the same letter is supposed to stand for the same number.

THE SIGNS, $+$, $-$, $\times$, $\div$.

3. Addition is denoted by the sign $+$ (read, *plus*), which indicates that the number following is to be added to that which precedes; thus, $3 + 2$ (that is, 3 plus 2) signifies that 2 is to be added to 3.

4. Subtraction is denoted by the sign $-$ (read, *minus*), which indicates that the number following is to be subtracted from that which precedes; thus, $7 - 4$ (that is, 7 minus 4) signifies that 4 is to be subtracted from 7.

These signs also denote the character, or quality, of numbers. Thus, those before which the plus ($+$) sign stands are called positive numbers; and those before which the minus ($-$) sign stands, negative numbers. This extension of meaning will be fully explained later.

5. Multiplication is denoted by the sign $\times$ (read, *into, times,* or *multiplied by*), which indicates that the number preceding is to be multiplied by that which follows; thus, 6×5 (that is, 5 times 6) signifies that 6 is to be multiplied by 5.

Between a figure and a letter, or between letters, the sign is usually omitted; thus, $6\,a\,b$ is the same as $6 \times a \times b$.

Sometimes the sign $\times$ is replaced by a point above the line; thus, $8 \cdot 6 \cdot 4$ is the same as $8 \times 6 \times 4$.

When figures are to be multiplied together, some sign for multiplication must always be employed; thus, 23 has already a meaning assigned to it,—namely, the sum of two tens and three units, or twenty-three,—and hence cannot stand for 2×3, or $2 \cdot 3$.

6. **Division** is denoted by the sign $\div$ (read, *divided by*), which indicates that the number preceding is to be divided by that which follows; thus, $9 \div 3$ (that is, 9 divided by 3) signifies that 9 is to be divided by 3.

Division is also indicated by the sign :, and by the fractional form; thus, $9 : 3$, $\frac{9}{3}$, and $9 \div 3$, all have for their value 3.

FACTORS AND POWERS.

7. A **Factor** is any one of several numbers, integral or fractional, which are to be multiplied together to form a *product*.

8. Any one or more of the factors which go to make up the product is called the **Coefficient** of the remaining factors. Thus, in $3\,a\,b\,c$, 3 is the coefficient of $a\,b\,c$, or $b\,c$ the coefficient of $3\,a$, or $3\,a\,b$ the coefficient of c, and so on.

A coefficient is called numerical, literal, or mixed, according as it is a numeral, a letter or letters, or a numeral and letters combined. The three cases above, taken in their order, illustrate this.

By coefficient, the numerical coefficient, together with the sign of the expression, is usually meant. If no figure is expressed, a unit is understood; thus, x is the same as $1\,x$.

9. The **Reciprocal** of a number is a unit divided by that number; thus, the reciprocal of 5 is $\frac{1}{5}$; of x, $\frac{1}{x}$.

10. A **Power** is the product obtained by repeating a number a given number of times.

11. An **Index**, or **Exponent**, is some number symbol, either positive or negative, integral or fractional, placed to the right, and a little above the number.

If the index is *positive* and *integral*, it indicates how many times the number enters as a factor into the power.

Thus, $2^4 = 2 \times 2 \times 2 \times 2 = 16$; read, 2 fourth power, or the fourth power of 2.

$a^2 = a \times a$; read, a second power, or a square.

$a^3 = a \times a \times a$; read, a third power, or a cube.

$a^n = a \times a \times a \; . \; . \; .$ to n factors; read, a nth power.

Exponents and coefficients must be carefully distinguished.

Thus, $x^4 = x \times x \times x \times x$

while $4x = x + x + x + x$

12. A **Root** is one of the equal factors into which a number may be resolved.

A root is indicated by the radical sign $\surd$, the initial letter of the word *radix*. The root index is written at the top of the sign, though the index denoting the second, or square, root is generally omitted. Thus,

$\sqrt{x}$; read, the second root, or the square root, of a.

$\sqrt[3]{x}$; read, the third root, or the cube root, of a.

$\sqrt[n]{x}$; read, the nth root of x.

ALGEBRAIC EXPRESSIONS.

13. An **Algebraic Expression** is a single number symbol, or a collection of such symbols, generally connected by algebraic signs.

14. The **Terms** of an algebraic expression are the parts which are connected by the signs + or −, the sign generally being considered as part of the term; thus, $3x + c - 7y$ is an algebraic expression of three terms, $3x$, $+c$, and $-7y$.

15. A **Monomial** is an algebraic expression which contains a single term; as, a, or $3x$, or $5bxy$.

16. A **Polynomial** is an algebraic expression which contains two or more terms; as, $x + y$, $3a + 4x - 7aby$, or $c + 2d - e + 5d$.

17. A **Binomial** is a polynomial of two terms; as, $3x + 3y$, or $x - y$.

18. A **Trinomial** is a polynomial of three terms; as, $3a + x - 6cd$.

19. A **Residual** is a binomial in which one term is plus and the other minus; as, $x - y$.

20. Algebraic expressions are sometimes classed as *simple* and *compound;* the former consisting of *one* term, the latter of *two* or *more.*

21. **Like Terms,** or **Similar Terms,** are those which do not differ, or differ only in their signs or coefficients; as, $4ax$, and $-3ax$. Other terms are *unlike* or *dissimilar;* as, $5cd$, and $4ab$.

22. The **Degree** of a term is denoted by the sum of the exponents of the literal factors; thus, $2a$ is of the first degree, and $6a^3x^4$ and $5a^2x^5$ are of the seventh degree.

23. Homogeneous Terms are those of the same degree; as, $5\,a\,x^2$, $3\,a\,b\,c$, and $x^2 z$.

24. A **Polynomial** is homogeneous if all its terms are of the same degree; thus, $4\,a\,x^2 + 5\,a\,b\,c + 3\,x\,y^2$ is a homogeneous polynomial.

SIGNS OF EQUALITY, INEQUALITY, GROUPING, ETC.

25. Equality is denoted by the sign $=$ (read, *equals*, or *is equal to*), which indicates that the number following it is equal to that which precedes it; thus, $1 = 100 cents, signifies that one dollar is equal to one hundred cents. Such a statement is called an equation; that portion which precedes the sign $=$ is called the *first member*, and that which follows, the *second member*.

26. Inequality is denoted by the sign $>$ or $<$ (read, *greater than*, *less than*), which indicates that the number standing at the vertex is less than the number standing at the opening of the two lines; thus, $6 < 8 < 9$ signifies that 8 is greater than 6, but less than 9.

27. The **Signs of Inference** are $\because$ (read, *because* or *since*), and $\therefore$ (read, *hence* or *therefore*).

$$\because\ 2^3 = 8, \quad \therefore\ \sqrt[3]{8} = 2.$$

28. The **Signs of Grouping** are the different forms of the Bracket ( ), [ ], { }, the Vinculum —, and the Bar |. They indicate that all the numbers included or connected are to be considered as a single number, and are to be subjected to the same operation; thus,

$$(a + b - c),\ [a + b - c],\ \{a + b - c\},\ \overline{a + b - c},\quad \left.\begin{array}{r} a \\ b \\ -c \end{array}\right|$$

indicate that a, b, and $-c$ are to be considered as one whole, and subjected to the same operation.

29. The **Sign of Continuation** is ..., or - - - (read, *and so on*); thus, $1 \cdot 2 \cdot 3 \ldots r$ signifies the product of the natural numbers from 1 to r inclusive, whatever the value of r.

The abbreviated form $\underline{|r}$ (read, *factorial* r) has the same signification.

30. The **Sign of Infinity** is ∞. In mathematics Infinity means a number which is greater than any assignable number.

31. Exercises in Translation from Algebraic into Common Language.

1. $a + b,\ a - b,\ a\,b,\ a \div b,\ a = b,\ a > b.$
2. $3^2,\ 3^3,\ 3^5\ 3^n,\ \sqrt{3},\ \sqrt[3]{3},\ \sqrt[5]{3},\ \sqrt[n]{3}.$
3. $5\sqrt{(6\,a^3 b^4)},\ \sqrt[3]{\dfrac{a\,b^4}{8\,x^3}},\ \sqrt{a} + \sqrt[3]{b}.$
4. $(a + b)^2,\ (a + b)^3 + 5\sqrt{a}.$
5. $(x + a)^2 - (x - a)^2 = 4\,a\,x.$
6. $(a + b) + (a - b) = 2\,a.$
7. $(a + b) - (a - b) = 2\,b.$
8. $(a + b)\,(a - b) = a^2 - b^2.$
9. $(m + 1)\,a + (n + 1)\,b = 44.$
10. $\left(x + \dfrac{1}{x}\right)\left(y + \dfrac{1}{y}\right) < z + \dfrac{1}{z}.$
11. $\dfrac{a + b}{x - y} - \dfrac{a - b}{x + y} = \sqrt{\dfrac{c + d}{m + n}}.$
12. $a^3 b^4 c^5 \times a^5 b^6 d^7 \div a^4 b^3 c^2.$
13. $a^5 - 5\,a^4 b + 10\,a^3 b^2 - 10\,a^2 b^3 + 5\,a\,b^4 - b^5.$
14. $\sqrt[3]{x - y} + \sqrt{5\,n} - (m + n)^2.$
15. $\dfrac{a^2 - b^2}{c - d} \times (x - y + z).$

32. Exercises in Translation from Common Language into Algebraic Language.

1. a is equal to b.

2. a is greater than c.

3. The sum of a and c is greater than their difference.

4. The quotient of a divided by c is less than their product.

5. The second power of a is equal to the cube of c increased by one.

6. The square root of a diminished by one is equal to the third root of c.

7. a prime, x sub naught, c nth power, b second sub two.

8. The sum of a, b, c, d, divided by their product equals what?

9. Five times the square root of a added to the cube of the sum of a and b equals zero.

10. Six times the cube of the sum of a and b divided by the sum of a and b equals six times the square of a added to twelve times the product of a and b, increased by six times the square of b.

33. Exercises in finding the Numerical Values of Literal Expressions.

To find the numerical value of an algebraic expression when the values of the letters are known, we must substitute the given values for the letters, and perform the operations indicated by the signs.

The numerical value of $8a - b^4 + c^2$, when $a = 4$, $b = 2$, and $c = 5$, is $8 \times 4 - 2^4 + 5^2 = 32 - 16 + 25 = 41$.

If $a = 16$, $b = 36$, $c = 9$, $d = 4$, $e = 1$, what are the values of

1. $2a - 3c + b - 4e - d.$

2. $4\sqrt{d} + 2\sqrt{b} - \sqrt{16e}.$

3. $\frac{4c}{d} - \frac{8ce}{b} + a.$

4. $\sqrt{c} + \sqrt{4d + c} - \sqrt[3]{4a}.$

5. Between the expressions $(a + b) - (c - e)$ and $(d + e)$, which of the signs $=$, $>$, or $<$ is correct?

6. Between the expressions $\frac{3a + 2e}{c - d}$ and $\sqrt{(a + c)d}$?

7. A boy expressed his age by saying, that if $a = 6$, $b = 5$, and $c = 7$, he was $\frac{6(a^2 + b^2 + c^2)}{a + ac}$ years old. How old was he?

8. A laborer's monthly pay was $\frac{1}{4}(a^2 + 2ab + b^2)$ dollars, and his annual expenses $54\left(\frac{a + b + c}{a^3}\right)$ dollars; did he save anything if $a = 3$, $b = 4$, $c = 5$?

34. Exercises in the Simplification of Numbers connected by Parentheses, and the Signs of Operation.

In reducing such expressions, the operations of multiplication and division must be performed before those of addition and subtraction.

Find the reduced values of the following expressions:

1. $10 + 15 \div 5 = ?$

2. $(10 + 15) \div 5 = ?$

3. $10 + 15 \div (5 \times 3) = ?$

4. $9 + \{(8 - 3) \div 5\} \times 2 = ?$

5. $9 + \overline{8 - 3 \div 5} \times 2 = ?$

6. $120 - (17 - 5) = ?$

7. $4 \div (2 \times 5) - 8 \times 2 \div 4 + 7 = ?$

8. $\sqrt[4]{1} + 3\sqrt{16} - \sqrt[3]{8} = ?$

35. Exercises in the use of Algebraic Language preparatory to the Solution of Simple Problems.

Should any difficulty arise in the attempt to answer the following questions, it is recommended that figures be substituted for the letters.

1. What number is greater than x by a?

2. By how much does x exceed 17?

3. How far can a man walk in a hours at the rate of 5 miles an hour?

4. If x is one factor of 12, what is the other?

5. If \$20 is divided among x persons, how much does each receive?

6. What dividend gives a as the quotient when 3 is the divisor?

7. By how much does $3a$ exceed a?

8. If 20 be divided into two parts and one part is x, what is the other?

9. The difference between two numbers is 13, and the smaller number is x; what is the greater?

10. If 100 contain x five times, what is the value of x?

11. What is the price, in cents, of 50 oranges, when x oranges cost 10 cents?

12. What is the cost of 40 books at x dollars each?

13. In x years a man will be 40 years old, what is his present age?

14. How old will a man be in a years if his present age is x years?

15. If the divisor is x, the quotient y, and the remainder z, what is the dividend?

16. If the divisor is $m + n$, the quotient $x + y$, and the remainder z, what is the dividend?

17. If a man is $x + y$ years old now, how old was he x years ago? How old y years ago?

18. How many hours will it take to walk x miles at 4 miles an hour?

19. How far can I walk in x hours at the rate of y miles an hour?

20. A bicyclist rides from Boston to Providence in two days. The first day he rides a hours at x miles an hour, the second day b hours at y miles an hour. Required the distance in miles from Boston to Providence. In feet.

AXIOMS.

36. The various operations performed upon equations are based upon certain self-evident truths called AXIOMS, of which the following are the most common:

1. If equals are added to equals the sums are equal.

2. If equals are subtracted from equals the remainders are equal.

3. If equals are multiplied by equals the products are equal.

4. If equals are divided by equals the quotients are equal.

5. Like powers and like roots of equals are equal.

6. The whole of a number is greater than any of its parts.

7. The whole of a number is equal to the sum of all its parts.

8. Numbers respectively equal to the same number, or equal numbers, are equal to each other.

37. Exercises in the Solution of Simple Problems.

The Solution of a Problem in Algebra consists,

1st. In reducing the statement to the form of an equation;

2d. In reducing the equation so as to find the value of the unknown numbers.

EXAMPLES.

1. The sum of two numbers is 90, and the larger is double that of the smaller. What are the numbers?

It is evident that if we knew the smaller number, by doubling it we should obtain the larger number. Suppose we let x equal the smaller number, then $2x$ must equal the larger; and, by the conditions of the problem, x the smaller number added to $2x$, the larger number, equals 90, or $3x = 90$. Therefore the smaller number is $\frac{1}{3}$ of 90, or 30, and $2x$, the larger number, is 60.

Expressed algebraically the process is as follows:

Let $x =$ the smaller number.
Then $2x =$ " larger "

By addition $x + 2x =$ their sum.
But $90 =$ " "

$$\therefore \quad x + 2x = 90$$
$$3x = 90$$
$$x = 30, \text{ the smaller number.}$$
$$2x = 60, \text{ " larger "}$$

2. A farmer has a horse, an ox, and a cow; the horse cost twice as much as the ox, and the ox twice as much as the cow, and all together cost \$350; how much did each cost?

Let $x =$ the number of dollars the cow cost.
Then $2x =$ " " " ox "
and $4x =$ " " " horse "

By addition $x + 2x + 4x =$ the number of dollars in the whole cost.

But $350 =$ the number of dollars in the whole cost.

$$\therefore \quad x + 2x + 4x = 350$$
$$7x = 350$$
$$x = 50$$
$$2x = 100$$
$$4x = 200$$

Therefore the cow cost \$50, the ox \$100, and the horse \$200.

3. Three men, A, B, and C, trade in company and gain \$300, of which B is to have twice as much as A, and C three times as much as A. Required the share of each.

Ans. A \$50, B \$100, C \$150.

4. In a certain garrison of 2700 men there are five times as many infantry and three times as many artillery as cavalry. How many are there of each?

5. A gentleman began trade with a certain sum of money, and continued in trade 3 years. At the end of the first and second years he found he had double what he had at the beginning of those years; but during the third year he lost as much money as he began business with, when, winding up his affairs, he found he had \$1800. How much money did he begin with? Ans. \$600.

6. A man has 3 horses which are together worth \$480, and their values are as the numbers 1, 2, and 3; what are their respective values?

Let x, $2x$, and $3x$ represent their respective values.

7. Divide 200 into three parts, in the proportion of 2, 3, and 5.

8. Two men are 180 miles apart and travel towards each other, one at the rate of 8 miles a day and the other at 10 miles a day. In how many days will they meet?

9. Four persons, A, B, C, and D, contributed towards a benevolent enterprise \$1800. B put in twice as much as A, C put in three times as much as B, and D put in as much as A, B, and C. How much did they each contribute?

Ans. A \$100, B \$200, C \$600, D \$900.

10. Required to find such a number that, if it be increased by 8, the result will be equal to 20.

Let $\quad x = $ the number.

Then $\quad x + 8 = 20$

But $\quad 8 = 8$

By subtraction $\quad x = 20 - 8$, or 12 (Ax. 2)

11. Required to find such a number that, if it be diminished by 8, the result will be equal to 20.

$$\begin{array}{lrl} \text{Let} & x & = \text{the number.} \\ \text{Then} & x - 8 & = 20 \\ \text{But} & 8 & = 8 \\ \hline \text{By addition} & x & = 20 + 8, \text{ or } 28 \quad (\text{Ax. 1.}) \end{array}$$

12. Find two numbers whose sum is 28, and whose difference is 6.

13. Find a number such that, if we double it and then add 20 to it, the result will be 140. Ans. 60.

14. Two persons agreed to give \$60 to a charity, one giving \$10 more than the other; what did each give?

15. Divide \$49 between A, B, and C, so that A may have \$11 more than B, and B \$7 more than C.

16. A man walks 10 miles, then travels a certain distance by train, and then twice as far by coach. If the whole journey is 100 miles, how far does he travel by train?

Ans. 30 miles.

17. A and B begin business, each with \$4500. B is unfortunate and loses yearly a certain amount, while A gains yearly the same sum until his money is double that of B's. What does A gain?

18. The sum of \$5500 was divided among 4 persons; the second received twice as much as the first, the third as much as the first and second, and the fourth as much as the second and third. How much did each receive?

19. Three men, A, B, and C, made a joint stock of \$3610; A put in a certain sum, B put in \$100 more than A, and C \$120 less than A. How much did each man put in?

Ans. A \$1210, B \$1310, C \$1090.

20. A person spent \$410 in buying sheep and cows. If each cow cost \$25, and each sheep \$5, and if the total number of animals bought was 42, how many of each did he buy?

CHAPTER II.

POSITIVE AND NEGATIVE NUMBERS.

38. WE speak of the temperature as being so many degrees above or below zero; of the navigator as having sailed so many degrees east or west of a given meridian; of the merchant as having gained or lost so much money; of an event as having occurred so many years before or after the Christian era.

Such opposition in the direction, character, or quality of numbers is indicated in algebra by the signs + and —. The numbers before which no sign, or the + sign, is placed, are called *positive numbers*, and those before which the — sign is placed, *negative numbers*.

Thus it will be seen that the signs + and —, beside their force as *signs of operation*, which has been explained, have a *merely* relative signification.

The two meanings assigned to these characters are always in accord, and no confusion can arise from regarding these signs in either sense, at pleasure. When, however, it is necessary to denote the character of a number and either addition or subtraction at the same time, we employ two signs, with the parenthesis, thus, $+(+9)$, $-(-9)$, the signs *within* indicating *character*.

The sign +, as a sign of character, is frequently omitted; and when neither the + sign nor the — sign is prefixed to a term, the + sign is to be understood.

The numbers representing the temperature above zero, north latitude, east longitude, assets, future time, etc., are

usually termed positive, and their opposites negative. But this is purely arbitrary, and there is nothing in the nature of things to prevent a reverse usage.

39. A clear conception of these numbers in all their relations can best be obtained through the device of a straight line with the zero point at its centre, and positive numbers extending to the right, and negative numbers to the left indefinitely, thus,

$$-\infty \ldots -1 \ldots -\tfrac{1}{2} \ldots 0 \ldots +\tfrac{1}{2} \ldots +1 \ldots +\infty,$$

a general series, which (imaginaries excepted) embraces all possible numbers; and thus,

$$-\infty \ldots -4, -3, -2, -1, 0, +1, +2, +3, +4, \ldots \infty,$$

a special series, embracing all possible integral numbers, which increase by one indefinitely from left to right, and decrease by one indefinitely from right to left. In these series every negative number is considered to be less than zero, and, in general, every number in these series is considered to be less than any number following it, and greater than any number preceding it, that is, *relatively* so.

In this *relative* sense, the phrases *greater than, less than, less than zero,* are to be understood, unless the contrary is expressed.

Arithmetic takes into account that part only of the series to the right of the zero, while algebra makes use of the whole series. From this it will be seen at once, that, in algebra, addition, subtraction, multiplication, and division, covering as they do both positive and negative numbers, must have a more extended signification than in arithmetic.

CHAPTER III.

ADDITION AND SUBTRACTION.

40. ADDITION in algebra is the process of finding the aggregate, or sum, of two or more algebraic expressions.

From the series in § 39 it is evident that the numbers added must be all positive, or all negative, or both positive and negative combined. When the numbers are all positive, the *algebraic sum* will be *positive* and equal in amount to the number of positive units; when they are all negative, the algebraic sum will be *negative* and equal in amount to the number of negative units; and when they are both positive and negative, the algebraic sum will be *positive* if the positive units are in the excess, and will be equal in amount to that excess, *negative* if the negative units are in the excess, and will be equal in amount to that excess, and *zero* if the sums of the positive and negative units are equal. Thus,

$$\left.\begin{array}{ll}(1) & +7+(+5)=+12 \\ (2) & -7+(-5)=-12 \\ (3) & +7+(-5)=+\ \ 2 \\ (4) & -7+(+5)=-\ \ 2 \\ (5) & +5+(-5)=\ \ \ \ \ \ 0\end{array}\right\} \quad . \quad . \quad . \quad (A)$$

41. In illustration of case (1), group (A), suppose a pencil be moved from the zero point, along the line of numbers, in § 39, second series, in the positive direction (that is, to the right) over seven spaces, then over five spaces, the distance from the starting point would be twelve spaces *to the right*, and would be represented by $+12\,s$, the *algebraic sum* of the distances moved.

In (2) the first movement would be seven spaces, the second five spaces, both in the negative direction (that is, to the left), and the distance from the starting point would be represented by $-12\,s$, the *algebraic sum* of the distances moved.

In (3) the first movement would be seven spaces to the right, the second five spaces to the left, and the distance from the starting point would be two spaces *to the right*, and would be represented by $+2\,s$, the *algebraic sum* of the distances moved.

In (4) the first movement would be seven spaces to the left, the second five spaces to the right, and the distance from the starting point would be two spaces *to the left*, and would be represented by $-2\,s$, the *algebraic sum* of the distances moved.

In (5) the first movement would be five spaces to the right, the second five spaces to the left, and the distance from the starting point would be zero, and would be represented by 0, the *algebraic sum* of the distances moved.

42. The *algebraic sum* is not then, as in arithmetic, the entire number of spaces moved, but the distance of the pencil, at the cessation of the movement, from the starting point.

And, in general, the algebraic sum of several numbers is the deviation of the result from zero, the positive units being counted *on*, or employed to affect the result, according to their number, in one way, and the negative units being counted *off*, or employed to affect the result, according to their number, in the opposite way.

43. Illustrative Problems.

1. Suppose a man to walk along a straight road 100 yards forward and then 70 yards backward, his distance from his starting point is 30 yards.

But if he first walks 70 yards forward, and then 100 yards backward, his distance from his starting point would be 30 yards, but *on the opposite side of his starting point.*

The corresponding algebraic statements would be

$$100 \text{ yd.} + (-\ \ 70 \text{ yd.}) = +\ 30 \text{ yd.}$$
$$70 \text{ yd.} + (-\ 100 \text{ yd.}) = -\ 30 \text{ yd.}$$

2. Suppose that I have a farm worth \$6000, and other property worth \$3000, and that I owe \$1000, then the net value of my estate is \$6000 + \$3000 + (– \$1000) = + \$8000. Again, suppose my farm is worth \$6000, and my other property \$4000, while I owe \$14000, then my net estate is worth \$6000 + \$4000 + (– \$14000) = – \$4000, that is, I am worth – \$4000, or, in other words, I owe \$4000 more than I can pay.

3. A boy played two games; in the first game he won 20 poiuts, and in the second he won –16 points (that is, he lost 16 points). How many did he win in all?

4. A thermometer indicated + 40° (40° above 0); it then fell 10°, then rose 30°. What temperature did it then indicate? Had it fallen, instead of risen, these last 30°, what would have been the temperature?

5. A ship sailed from the equator due north 40 miles the first day of her voyage, the second day 20 miles due north, the third day 80 miles due south. What was her latitude at the end of the third day? In this example, which contains the greater number of units, the algebraic or the arithmetical sum?

SUBTRACTION.

44. Subtraction consists in finding the *difference* between two numbers. This *difference* is the number of units which lie between the two numbers, or is what must be added to the subtrahend to produce the minnend.

It follows, that subtraction is the inverse of addition, and must not be considered a distinct process from addition.

45. From (3), (4), (1), (2), of group (A), § 40, considering the 7's minuends, and the 5's subtrahends, we produce by addition, that is, by determining what must be added to the subtrahend to produce the minuend, the following:

$$\left.\begin{array}{ll}(1) & +7-(-5)=+12 \\ (2) & -7-(+5)=-12 \\ (3) & +7-(+5)=+\ \ 2 \\ (4) & -7-(-5)=-\ \ 2\end{array}\right\} \quad . \ . \ . \quad (B)$$

46. Wherever the minuend and subtrahend may be situated in the series (§ 39), no spaces, or a certain number of spaces, will lie between them, and this number, with the appropriate sign, will represent their algebraic difference.

If the movement is to the *right*, in going from the subtrahend to the minuend, the + sign would be prefixed; if to the *left*, the − sign.

In the illustration of case (1), group (B), the movement would be from a point five spaces to the left of zero to a point seven spaces to the *right;* hence the algebraic difference, $+12\,s$.

In (2), the movement would be from a point five spaces to the right of zero to a point seven spaces to the *left;* hence the algebraic difference, $-12\,s$.

In (3), the movement would be from a point five spaces to the right of zero to a point seven spaces to the *right;* hence the algebraic difference, $+2\,s$.

In (4), the movement would be from a point five spaces to the left of zero to a point seven spaces to the *left;* hence the algebraic difference, $-2\,s$.

47. From groups (B) and (A), (2) and (1), it follows that subtracting a positive number is equivalent to adding an equal negative number, and subtracting a negative number is equivalent to adding an equal positive number.

Therefore, to subtract one number from another, *change the sign of the subtrahend and proceed as in addition.*

48. Illustrative Problems.

1. Suppose I am worth \$9000; it matters not whether a thief steals \$4000 from me, or a rogue having the authority involves me in debt \$4000 for a worthless article; for in either case I shall be worth only \$5000. The thief *subtracts* a *positive* quantity; the rogue *adds* a *negative* quantity.

The corresponding algebraic statements are,

$$\$9000 - (+\$4000) = +\$5000$$

and

$$\$9000 + (-\$4000) = +\$5000$$

2. Augustus was born B.C. 63, and died A.D. 14. How old was he when he died?

3. If A has \$8000 and no debts, and B has no property but owes \$4000, how much better off is A than B?

4. The longitude of Paris is 2° E., that of Boston 71° W. What is their difference of longitude?

5. The longitude of San Francisco is 122° W., that of Boston 71° W. What is their difference of longitude?

ADDITION AND SUBTRACTION OF ALGEBRAIC LITERAL EXPRESSIONS.

49. Let a and b stand for any positive numbers whatever, (1) and (3) in group (A), and (3) and (1) in group (B) will become:

$$\left.\begin{array}{ll}(1) & +a+(+b)=a+b\\(2) & +a+(-b)=a-b\\(3) & +a-(+b)=a-b\\(4) & +a-(-b)=a+b\end{array}\right\}\quad .\quad .\quad .\quad .\quad (C)$$

50. To prove the above laws true for all negative values, let $c = -b$, where b is any positive quantity; then c is any negative quantity, and we have

$$+c = +(-b) = -b$$
$$-c = -(-b) = +b$$

Substituting $+c$ for $-b$, and $-c$ for $+b$ in (1), (2), (3), (4), we get

$$
\begin{aligned}
a + (-c) &= a - c\\
a + (+c) &= a + c\\
a - (-c) &= a + c\\
a - (+c) &= a - c
\end{aligned}
$$

the same laws as before, although in a different order. Hence the laws expressed in (C) are true for *all values* of b.

51. This proof shows that a letter may stand for any value whatever, and that a letter preceded by the $+$ sign, for example $+b$, is not necessarily positive.

Such terms as $+b$, $-a$, are called positive and negative terms, because of their outward form, though not so necessarily. The signs before them indicate what is to be done with them when they enter into operations, but nothing as to their reduced, or ultimate value.

52. The addition of Algebraic Literal Expressions may be conveniently presented under four heads.

Case I.

53. To find the Sum of Monomials when they are Similar and have Like Signs.

1. John has 5 apples, Thomas 8 apples, and Frank 3 apples; how many apples have they all?

$$
\left.\begin{array}{r} 5 \text{ apples,}\\ 8 \text{ apples,}\\ 3 \text{ apples,}\\ \hline 16 \text{ apples,}\end{array}\right\}
\begin{array}{c}\text{or, letting } a\\ \text{represent}\\ \text{one apple,}\end{array}
\left\{\begin{array}{r} 5\,a\\ 8\,a\\ 3\,a\\ \hline 16\,a\end{array}\right.
$$

It is evident that just as 5 apples and 8 apples and 3 apples added together make 16 apples, so $5\,a$ and $8\,a$ and $3\,a$ added together make $16\,a$.

In the same way $-5\,a$ and $-8\,a$ and $-3\,a$ are equal together to $-16\,a$.

Therefore, when the monomials are similar, and have like signs, we have the following

Rule.

Add the coefficients, and to their sum annex the common letter or letters, and prefix the common sign.

(2.)	(3.)	(4.)	(5.)	(6.)	(7.)
$3\,a\,x$	$5\,a^2$	x	$5\,y$	$-\,5\,x^3$	$-\,5\,b\,y$
$9\,a\,x$	$3\,a^2$	$5\,x$	$10\,y$	$-\,4\,x^3$	$-\,4\,b\,y$
$7\,a\,x$	$8\,a^2$	$8\,x$	y	$-\,8\,x^3$	$-\;\;b\,y$
$4\,a\,x$	$4\,a^2$	$4\,x$	$3\,y$	$-\,6\,x^3$	$-\,3\,b\,y$
$23\,a\,x$		$18\,x$		$-\,23\,x^3$	

8. What is the sum of $a\,x^2$, $4\,a\,x^2$, $5\,a\,x^2$, and $3\,a\,x^2$?
Ans. $13\,a\,x^2$.

9. What is the sum of $4\,b\,x$, $8\,b\,x$, $7\,b\,x$, and $b\,x$?

10. What is the sum of $4\,x\,y$, $5\,x\,y$, $10\,x\,y$, and $9\,x\,y$?

11. What is the sum of $-\,8\,x\,z$, $-\,x\,z$, $-\,3\,x\,z$, and $-\,x\,z$?
Ans. $-\,13\,x\,z$.

12. What is the sum of $-\,3\,b$, $-\,4\,b$, $-\,5\,b$, and $-\,3\,b$?

13. What is the sum of $-\,a\,b\,c\,d$, $-\,4\,a\,b\,c\,d$, $-\,7\,a\,b\,c\,d$, and $-\,5\,a\,b\,c\,d$?

Case II.

54. To find the Sum of Monomials when they are Similar and have Unlike Signs.

1. A man earns 8 dollars one week, and the next week earns nothing and spends 5 dollars, and the next week earns 3 dollars, and the fourth week earns nothing and spends 4 dollars; how much money has he left at the end of the fourth week?

If what he earns is indicated by +, then what he spends will be indicated by —, and the example will appear as follows:

$$\left.\begin{array}{r} +\ 8 \text{ dollars,} \\ -\ 5 \text{ dollars,} \\ +\ 3 \text{ dollars,} \\ -\ 4 \text{ dollars,} \\ \hline +\ 2 \text{ dollars,} \end{array}\right\} \begin{array}{c} \text{or, letting } d \\ \text{represent} \\ \text{one dollar,} \end{array} \left\{\begin{array}{r} +\ 8\ d \\ -\ 5\ d \\ +\ 3\ d \\ -\ 4\ d \\ \hline +\ 2\ d \end{array}\right.$$

Earning 8 dollars and then spending 5 dollars, the man would have 3 dollars left; then earning 3 dollars, he would have 6 dollars; then spending 4 dollars, he would have left 2 dollars; or he earns in all 8 dollars $+$ 3 dollars $=$ 11 dollars; and spends 5 dollars $+$ 4 dollars $=$ 9 dollars; and therefore has left the difference between 11 dollars and 9 dollars $=$ 2 dollars; hence the sum of $+ 8\,d$, $- 5\,d$, $+ 3\,d$, and $- 4\,d$, is $+ 2\,d$.

Therefore, when the terms are similar, and have unlike signs, we have the following

Rule.

Find the difference between the sum of the coefficients of the positive terms, and the sum of the coefficients of the negative terms, and to this difference annex the common letter or letters, and prefix the sign of the greater sum.

(2.)	(3.)	(4.)	(5.)
$4\,x\,y$	$5\,y^2$	$14\,a\,b\,c^2$	$x^2\,y$
$x\,y$	$-\ 2\,y^2$	$7\,a\,b\,c^2$	$-\ 15\,x^2\,y$
$-\ 6\,x\,y$	$8\,y^2$	$-\ \ a\,b\,c^2$	$11\,x^2\,y$
$8\,x\,y$	$-\ 4\,y^2$	$-\ 9\,a\,b\,c^2$	$-\ 13\,x^2\,y$
$-\ 2\,x\,y$	$14\,y^2$	$4\,a\,b\,c^2$	$25\,x^2\,y$
$5\,x\,y$		$15\,a\,b\,c^2$	

(6.)	(7.)
$26\,x\,y\,z$	$9\,(x+y)$
$-\ 60\,x\,y\,z$	$-\ 5\,(x+y)$
$20\,x\,y\,z$	$8\,(x+y)$
$-\ 68\,x\,y\,z$	$-\ 4\,(x+y)$
$9\,x\,y\,z$	$-\ 2\,(x+y)$
$-\ 73\,x\,y\,z$	$6\,(x+y)$

8. Find the sum of $9x^2y^2$, $-15x^2y^2$, $18x^2y^2$, and $-x^2y^2$.

9. Find the sum of $9(x+y)$, $10(x+y)$, $-2(x+y)$, and $(x+y)$. Ans. $18(x+y)$.

10. Find the sum of $-ax^2$, $+ax^2$, $+11ax^2$, $+26ax^2$, and $-14ax^2$.

11. Find the sum of $28abc$, $-34abc$, $-150abc$, $27abc$, and $-13abc$.

12. Find the sum of a^3x^3, $-15a^3x^3$, $18a^3x^3$, $-a^3x^3$, $23a^3x^3$, and $-a^3x^3$.

13. Find the sum of $19(a+b)$, $-(a+b)$, $(a+b)$, and $-14(a+b)$. Ans. $5(a+b)$.

Case III.

55. To find the Sum of any Monomials.

From (2) in group (C) it follows that

$$a+(-b)=a-b$$

and that the sum of a, $-b$, and $-c$, or

$$a+(-b)+(-c)=a-b-c.$$

No further reduction is possible, and therefore, to add dissimilar monomials, *write them one after the other, each with its proper sign.*

All algebraic expressions can be so written, and the result, without further reduction, is sometimes called an *algebraic sum.*

56. It should be remarked that

$$a+b+c=a+c+b=b+a+c=\text{etc.};$$

that is, the sum of any number of algebraic expressions is *independent of their order.* This is called the Commutative Law of Addition.

57. Further,

$$+(a+b)=+a+b.$$

For, to put a and b together, and then add the result to what has gone before, is the same as to add both a and b to what has gone before. Similarly,

$$a + (b + c) = (a + b) + c$$

that is, the sum of any number of algebraic expressions is *independent of the mode of grouping them.* This is called the Associative Law of Addition.

58. It follows that

the sum of $5a$ and $6b$ is neither $11a$ nor $11b$, and can only be expressed in the form of $5a + 6b$, or $6b + 5a$; and the sum of $5a$ and $-4b$ is $5a - 4b$, or $-4b + 5a$. In finding the sum of $5a$, $6b$, $5a$, and $-4b$, the a's can be added together by Case I., and the b's by Case II., and the two results arranged according to §§ 56, 57; thus, $5a + 6b + 5a - 4b = 6b + 5a - 4b + 5a = 6b - 4b + 5a + 5a = 2b + 10a$, or $10a + 2b$, regardless of the order of the terms.

1. Find the sum of $7d$, $-4b$, x, $3y$, $8x$, $-3b$, $3bc$, $5d$, $5x$, $7b$, $4x$, and $-3bc$.

$$\begin{array}{rrrrr}
7d & -4b & +x & +3y & +3bc \\
5d & -3b & +8x & & -3bc \\
 & +7b & +5x & & \\
 & & +4x & & \\
\hline
12d & & +18x & +3y &
\end{array}$$

For convenience, similar terms are written under each other; then by Case I. the first column at the left is added; the second by Case II., and so on; $+3bc$ and $-3bc$ cancel.

Therefore, to find the sum of any monomials, we have the following

Rule.

Add the similar terms according to Cases I. and II., and write after these results, in any order, the dissimilar ones, each with its proper sign.

2. Find the sum of $2a$, $-3b$, $+c$, $+b$, $-2c$, $+3d$, $-a$, $+3c$, and $-2d$.

3. Find the sum of $4x$, $-7a$, $+3y$, $-4b$, $+3z$, $+6a$, $-y$, $+4b$, $-2z$, $-2y$, $+4a$, $+8b$, $-z$, $-3a$, $-8b$, and $-10c$.

4. $7+(-9)+(-1)+(0)+(+5)+(-6)=?$

5. $2a^2+(-3b^2)+(+4b^2)+(+9a^2)+(-b^2)=?$

6. $a+(-2b)+(+3a)-(-b)+(-4a)+(+4b)=?$

Case IV.

59. To find the Sum of Polynomials.

Polynomials are groups of monomials, and hence everything necessary to a complete understanding of their addition has been explained in the three foregoing cases.

60. If in the various operations with polynomials the monomials composing them be first arranged according to the ascending or descending powers of some letter, symmetry will be secured, and a consequent less liability to error, in the operations to follow.

The polynomials,

$$x^3+4x^2+5x+2x^0, \quad \text{and} \quad 2x^0+5x+4x^2+x^3,$$

are said to be *arranged* according to the descending and ascending powers of x, respectively.

We have therefore for the addition of polynomials the following

Rule.

Write similar terms under each other, find the sum of each column, and connect the several sums with their proper signs.

1. Find the sum of $3a - 7b + 2c$, $6a - b + 5c$, $-4a + 3b - 8c$.

$$\begin{array}{r} 3a - 7b + 2c \\ 6a - b + 5c \\ -4a + 3b - 8c \\ \hline 5a - 5b - c \end{array}$$

(2.)

$$\begin{array}{rrrr} -2b^3 & +3ab^2 & & +a^3 \\ & -ab^2 & +5a^2b & -3a^3 \\ 5b^3 & & & +8a^3 \\ & +ab^2 & +9a^2b & -2a^3 \\ \hline 3b^3 & +3ab^2 & +14a^2b & +4a^3 \end{array}$$

(3.)

$$\begin{array}{rrrr} 3x^3 & & +5\sqrt{x} & +7 \\ & -9x & +2\sqrt{x} & -8 \\ -2x^3 & +4x & +3\sqrt{x} & \\ \hline x^3 & -5x & +10\sqrt{x} & -1 \end{array}$$

Add the following:

4. $4x - 2y + 1$, $-3x + 2 - y$, $x + 3y + 3$.

5. $x^3 - 2x^2y - 2xy^2$, $x^2y - 3xy^2 - y^3$, $3xy^2 - 2y^3 - x^3$.

6. $a + b - c + 7$, $-2a - 3b - 4c + 9$, $3a + 2b - 16 + 5c$.

7. $16x^3 - 12 - 2x$, $11 - 11x^3 - 7x^2$, $9x^3 - x + 1 - x^2$, $5 + 7x + 8x^2$, $-x^3 - 9x$.

8. $11x^4 - 2xy^3$, $3x^3y - 2x^2y^2 + 7xy^3 - 8y^4$, $8x^4 - 7x^3y + y^4$, $-12x^4 + 4x^3y + 5x^2y^2$, $-3x^2y^2 - 5xy^3 + 7y^4$.

9. $3ab^2 - 2b^3 + a^3$, $5a^2b - ab^2 - 3a^3$, $8a^3 + 5b^3$, $9a^2b - 2a^3 + ab^2$. Ans. $4a^3 + 14a^2b + 3ab^2 + 3b^3$.

10. $a^3 - 2\,b^3 - 11\,a^2\,b - 4\,a\,b^2 - b\,c^2$, $4\,b^3 - a\,b\,c + 6\,c^3 + 9\,a^2\,b - a\,c^2 + a\,b^2$, $c^3 - 2\,b^3 - a^3 + 4\,b\,c^2 + 3\,a\,b^2$, $2\,a^2\,b - 3\,b\,c^2 - 7\,c^3 + 3\,a\,b\,c$. Ans. $2\,a\,b\,c - a\,c^2$.

11. $3\,x^2 - 10\,y^2 + 5\,z^2 - 7\,y\,z$, $-x^2 + 4\,y^2 - 10\,z^2 + 3\,x\,y$, $z^2 + 11\,y\,z + 8\,x\,z - 2\,x\,y$, $4\,z^2 - 4\,y\,z + x\,z$, $-2\,x^2 + 6\,y^2 - 9\,x\,z - x\,y$.

12. $4\,a^5 + 12\,a^3 - a - 10$, $6\,a^4 - a^3 + 2\,a^2 - 7$, $9\,a^2 - 3\,a^5 + 4\,a$, $11\,a - 2\,a^4 - a^5 + 9$, $4\,a^3 - a^4 - 5$.

13. $11\,x^5 - 9\,x^4 + 1$, $2\,x^5 - 3\,x^4 - 2\,x$, $x^4 - x + 12$, $x^5 + 4\,x - 3$, $8\,x^5 + 7\,x^4$.

14. $x^2\,y\,z - x^3\,z - 2\,x\,z^3 - 7\,z^4$, $3\,y^4 - x^4 - 2\,x^2\,y^2 - 3\,x^3\,z - 4\,x\,y^3$, $-2\,x^4 - y^4 - 10\,x\,y^2\,z + 4\,x^2\,z^2 + 4\,x\,z^3$, $3\,x^4 + 8\,x\,y\,z^2 - 4\,x^2\,z^2 + 3\,y^4 + 7\,z^4 - x^2\,y\,z$, $4\,x^3\,z + 2\,y^4 + 2\,x^2\,y^2 + 4\,x\,y^3 + 10\,x\,y^2\,z$, $-7\,y^4 - 2\,x\,z^3 - 8\,x\,y\,z^2$.

15. $\frac{1}{2}\,a - \frac{1}{3}\,b$, $-a + \frac{2}{3}\,b$, $\frac{3}{4}\,a - b$.

16. $\frac{2}{3}\,x^2 + \frac{1}{3}\,x\,y - \frac{1}{4}\,y^2$, $-x^2 - \frac{2}{3}\,x\,y + 2\,y^2$, $\frac{2}{3}\,x^2 - x\,y - \frac{5}{4}\,y^2$.

17. $3\,a^2 - \frac{2}{5}\,a\,b - \frac{1}{2}\,b^2$, $-\frac{3}{2}\,a^2 + 2\,a\,b - \frac{2}{3}\,b^2$, $-\frac{2}{3}\,a^2 - a\,b + b^2$.

18. $\frac{1}{2}\,a^3 - 2\,a^2\,b - \frac{3}{2}\,b^3$, $\frac{3}{2}\,a^2\,b - \frac{3}{4}\,a\,b^2 + 2\,b^3$, $-\frac{3}{2}\,a^3 + a\,b^2 + \frac{1}{2}\,b^3$.

19. $8\,y\,\sqrt{x} - 3\,x\,\sqrt{y} + 5$, $\sqrt{x\,y} + 3\,x\,\sqrt{y} + 3$, $2\,y\,\sqrt{x} - \sqrt{x\,y} - 6$, $7\,y\,\sqrt{x} - 4\,x\sqrt{y} - 3$, $1 + 7\,x\,\sqrt{y} - 2\,y\,\sqrt{x}$.

20. $4\,\sqrt{a\,b} - 5\,\sqrt{c\,d} + 6\,y\,\sqrt{x}$, $-3\,\sqrt{c\,d} - 5\,\sqrt{a\,b} + 2\,y\,\sqrt{x}$, $+7\,y\,\sqrt{x} - 3\,\sqrt{a\,b} + 8\,\sqrt{c\,d}$, $3\,y\,\sqrt{x} + 4\,\sqrt{a\,b} + 2\,\sqrt{c\,d}$.

SUBTRACTION OF ALGEBRAIC LITERAL EXPRESSIONS.

61. From group (C) we learn that, in general, *the subtraction of a positive number is equivalent to adding an*

equal negative number, and the subtraction of a negative number is equivalent to adding an equal positive number.

Therefore, for the subtraction of Monomials and Polynomials, we have the following

Rule.

Change the sign of each term of the subtrahend from + to —, or — to +, or suppose each to be changed, and then proceed as in addition.

	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)
Min.	9	9	9	9	9	9	9
Sub.	9	6	3	0	-3	-6	-9
Rem.	0	3	6	9	12	15	18

In examples 1–7, the minuend remaining the same while the subtrahend becomes in each 3 less, the remainder in each is 3 greater than in the preceding.

	(8.)	(9.)	(10.)	(11.)	(12.)	(13.)	(14.)
Min.	9	6	3	0	-3	-6	-9
Sub.	9	9	9	9	9	9	9
Rem.	0	-3	-6	-9	-12	-15	-18

In examples 8–14, the minuend in each becoming 3 less while the subtrahend remains the same, the remainder in each is 3 less than in the preceding.

	(15.)	(16.)	(17.)	(18.)	(19.)	(20.)	(21.)
Min.	9	6	3	0	-3	-6	-9
Sub.	9	6	3	0	-3	-6	-9
Rem.	0	0	0	0	0	0	0

In examples 15–21, both minuend and subtrahend decreasing by 3, the remainder remains the same.

	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)
Min.	$37x$	$28axy$	$-14ab$	$-19c$	$48xyz$	$-638bcd$
Sub.	$11x$	$-7axy$	$5ab$	$-7c$	$-26xyz$	$29bcd$
Rem.	$26x$	$35axy$	$-19ab$	$-12c$		

	(7.)	(8.)	(9.)	(10.)	(11.)	(12.)
Min.	$20x$	$-11abc$	$9xy$	$25z$	$-30xy$	$28cd$
Sub.	$46x$	$30abc$	$+15xy$	$20z$	$59xy$	$-560cd$
	$-26x$	$-41abc$	$-6xy$	$5z$		

(13.)

$$\begin{array}{ll} \text{Min.} & 9x - 11y - 3z \\ \text{Sub.} & \underline{5x + 2y + 4z} \\ & 4x - 13y - 7z \end{array}$$

(14.)

$$\begin{array}{r} 18a + 22b - 33c \phantom{{}+17d} \\ \underline{-40b + 5c - 17d} \\ 18a + 62b - 38c + 17d \end{array}$$

15. $-7-(-1)=?$

16. $-3ab-(+ab)=?$

17. $3a^2b-(+a^2b)+(+5a^2b)=?$

18. $27a^2-(+b^2)+(+9b^2)-(-2a^2)-(+b^2)=?$

19. From $-3a$ take $+8a$.

20. From $10b^2$ take $-7b^2$.

21. From $-12x^2y$ take $16x^2y$.

22. From $20abc$ take $-4abc$.

23. From $25a-16b-18c$ take $4a-3b+15c$.

24. From $7a^5+11a^4b+3a^2b^3-4ab^4-b^5$ take $5a^3b^2-4a^5+2a^2b^3+11a^4b+b^5$.

25. From $3ab+5cd-4ac-6bd$ take $3ab+6cd-3ac-5bd$.

26. From $10a^2b+8ab^2-8a^3b^3-b^4$ take $5a^2b-6ab^2-7a^3b^3$.

27. Subtract x^3-x^2+x+1 from x^3+x^2-x+1.

28. Subtract $b^3 + c^3 - 2abc$ from $a^3 + b^3 - 3abc$.

29. Subtract $1 - x + x^5 - x^4 - x^3$ from $x^4 - 1 + x - x^2$.

30. Subtract $a^3 - 3a^2b + 3ab^2 + b^3$ from $a^3 + 3a^2b + 3ab^2 + b^3$.

31. From $5\sqrt{xy} + 2x - 7a$ take $3\sqrt{xy} - x + 2a$.

32. From $\frac{1}{3}x - \frac{3}{2}y - \frac{1}{6}$ take $-\frac{1}{2}x + \frac{2}{3}y - \frac{1}{6}$.

33. From $\frac{1}{8}a^3 - 2ax^2 - \frac{1}{3}a^2x$ take $\frac{1}{3}a^2x + \frac{1}{4}a^3 - \frac{3}{2}ax^2$.

34. From $5x^3 + 3x - 1$ take the sum of $2x - 5 + 7x^2$ and $3x^2 + 4 - 2x^3 + x$.

35. Subtract $5x^2 + 3x - 1$ from $2x^3$, and add the result to $3x^2 + 3x - 1$.

36. Add the sum of $2y - 3y^2$ and $1 - 5y^3$ to the remainder left when $1 - 2y^2 + y$ is subtracted from $5y^3$.

37. What must be added to $5x^2 - 7x + 2$ to produce $7x^2 - 1$?

38. What must be subtracted from $3a - 5b + c$ to leave $2a - 4b + c$?

39. From what must $11a^2 - 5ab - 7bc$ be subtracted to give the remainder $5a^2 + 7ab + 7bc$?

40. To what must $7x^3 - 6a^2 - 5x$ be added to make $9x^3 - 6x - 7x^2$?

41. If $3x^2 - 7x + 5$ be subtracted from zero, what will be the result?

42. To what must $5ab - 11bc - 7ca$ be added to produce zero?

43. Subtract $3x^3 - 7x + 1$ from $2x^2 - 5x - 3$, then subtract the difference from zero, and add this last result to $2x^2 - 2x^3 - 4$.

44. Subtract $3x^2 - 5x + 1$ from unity, and add $5x^2 - 6x$ to the result.

45. Take the sum of $x^3 + 3x - 2$, $2x^3 + x^2 - x + 5$, and $4x^3 + 2x^2 - 7x + 4$, from the sum of $2x^3 + 9x$ and $5x^3 + 3x^2$.

46. From the sum of $12x^5 + 4xy^4 + y^5$, $2x^5 - 4x^4y - xy^4 + 3y^5$, and $6x^4y + 2x^2y^3 - 3xy^4$, take the sum of $6x^5 + 2x^2y^3 - y^5$, $x^5 - 2x^4y + x^3y^2 + 2y^5$, and $6x^5 + 4x^4y - 2x^3y^2 + 3y^5$. Ans. $x^5 + x^3y^2$.

NOTE. — In examples like the last, where several polynomials are to be combined, some by addition and some by subtraction, the result can best be obtained by writing down the polynomials to be subtracted with their signs reversed under those to be added, and then finding the sum of all, thus making use of but a *single* process.

Removal and Introduction of Brackets.

62. The addition or subtraction of a polynomial may be indicated by enclosing the polynomial in a bracket and prefixing the sign $+$ for addition, and the sign $-$ for subtraction. The polynomial $d + e - f$ added to the polynomial $a + b + c$, that is, $a + b + c + (d + e - f)$, equals *by the rules of addition* $a + b + c + d + e - f$; and the polynomial $d + e - f$ subtracted from the polynomial $a + b + c$, that is, $a + b + c - (d + e - f)$, equals *by the rules of subtraction* $a + b + c - d - e + f$.

Therefore, for the removal of brackets we have the following

Rule.

When the bracket with the plus sign before it is removed, the included terms must be rewritten without change of sign; but when a bracket with the minus sign before it is removed, the included terms must be rewritten with change of sign.

63. When there are several brackets, they may be removed *one at a time*.

Thus,

$$
\begin{aligned}
& a - 2b - [4a - 6b - \{3a + (5a - \overline{2b - 3a})\}] \\
&= a - 2b - [4a - 6b - \{3a + (5a - 2b + 3a)\}] \\
&= a - 2b - [4a - 6b - \{3a + 5a - 2b + 3a\}] \\
&= a - 2b - [4a - 6b - 3a - 5a + 2b - 3a] \\
&= a - 2b - 4a + 6b + 3a + 5a - 2b + 3a \\
&= 8a + 2b \text{ by adding similar terms.}
\end{aligned}
$$

In the above process the vinculum was removed first, and then the brackets in succession, beginning with the inner one.

If we remove the outer bracket first, the work will appear as follows:

$$
\begin{aligned}
& a - 2b - [4a - 6b - \{3a + (5a - \overline{2b - 3a})\}] && (1) \\
&= a - 2b - 4a + 6b + \{3a + (5a - \overline{2b - 3a})\} && (2)
\end{aligned}
$$

As the $+$ sign now appears before the next two brackets, these might have been omitted at once from (2) without further change of signs, and at the same time the *vinculum* over $2b - 3a$, and the final expression obtained at once. Thus from (1) we have at once

$$a - 2b - 4a + 6b + 3a + 5a - 2b + 3a = 8a + 2b.$$

It is recommended that the more advanced students begin with the outermost bracket, as the shorter of the two methods.

Remove the brackets, and reduce each of the following expressions to its simplest form.

1. $a + (-b + c - d + e)$.
2. $a - (b - c + d - e)$.
3. $3a - 5b + 2c + (2a + 3b - c)$.
4. $a - [2a - \{3b - (4c - 2a)\}]$.
5. $-\{-[-(a - \overline{b - c})]\}$.
6. $-(-(-(-x))) - (-(-y))$.
7. $8x - \{16y - [3x - (12y - x) - 8y] + x\}$.
8. $-[a - \{a + (x - a) - (x - a) - a\} - 2a]$.
9. $5 - [4 + \{5 - (4 + \overline{5 - 4})\}]$.

10. $3x - \{2y + (5x - \overline{3x + y})\}$.

11. $\{2x - (5y - \overline{3z + 7})\} - [4 + \{x - (3y + 2z + 5)\}]$.

12. $1 - [1 - \{-1 - (1 - 1) - 1\} - 1]$.

13. $1 - (2 - (3 - (4 - (9 - (10 - 11)))))$.

14. $a - [5b - \{a - (5c - \overline{2c - b}) + 2a - (a - 2b - c)\}]$.

15. $(3b^4 + 8cyz) + \{(7 - b^4) - (7 - 8b^4)\} - (8cyz + 6b^4)$.

16. $\{(3x^2y^2z^2 - 2yz^2) - (xyz - yz^2)\} - (2x^2y^2z^2 - xyz) - (x^2y^2z^2 - yz^2)$.

17. $6m + \{4m - [8n - (2m + 4n) - 22n] - 7n\} - \{7n + [9m - (3n + 4m) + 8n] + 6m\}$.

18. $a - \{5b + (c - 3a) + 4b\} + [6a - (3b + 2c)]$.

19. $a - 2b - [4a - 6b - \{3a - c + (5a - 2b - \overline{3a - c + 2b})\}]$.

20. $-[-2x - \{3y - (2x - 3y) + (3x - 2y)\} + 2x]$.

64. Conversely, from § 62, we have for the introduction of brackets the following

Rule.

When the bracket introduced is preceded by the plus sign, all the terms enclosed must be written without change of sign; but when the bracket is preceded by a minus sign, all the terms enclosed must be written with change of sign.

According to this rule, a polynomial can be written in a variety of ways.

Thus,
$$\begin{aligned} & x^3 - 3x^2y + 3xy^2 - y^3 \\ &= x^3 - (3x^2y - 3xy^2 + y^3) \\ &= x^3 - 3x^2y - (-3xy^2 + y^3) \\ &= x^3 - y^3 - (3x^2y - 3xy^2) \end{aligned}$$

Place in brackets, with the sign $-$ prefixed, without changing the value of the expression:

1. The last two terms of

$$a + b - c + d.$$

2. The last three terms of

$$x - y - z - u.$$

3. The first three and the last three terms of

$$-3a - 4b + 2c - 3d + e - f.$$

4. The last four terms of

$$-2a - 3c - d + 2e + d.$$

Bracket, without change of value, the following expressions two together in their order, then three together in their order, with the minus sign before the bracket in each case:

5. $a - b + c - d + f - g.$

6. $-2c + 3d - e + 4f + 3a - 7b.$

7. $4a + 6b - 5c + 2d - 3e + 3f.$

8. Place, without change of value, $a - b + c - d + e - f$ in the following set of brackets, so that $e - f$ shall stand in the innermost bracket, $c - d$ in the middle, and $a - b$ nearest the left.

$$-[\qquad -\{\qquad -(\qquad)\}].$$

Rewrite the following polynomials, so that, without change of value, they shall be composed of binomial instead of monomial terms:

9. $a - b - c + d + e - f.$

10. $a^2 - ab - ac + b^2 - bc + c^2.$

11. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$

CHAPTER IV.

MULTIPLICATION.

65. MULTIPLICATION is a short method of finding the sum of the repetitions of a number, or of the repetitions of a certain part of that number.

The Laws of Multiplication.

66. In Algebra, as in Arithmetic, $a\,b = b\,a$, and so for any number of factors, as $a\,b\,c = a\,c\,b = b\,c\,a$, or the product of any number of factors is *independent of their order.*

This is called the Commutative Law of Multiplication.

67. Moreover $a\,(b\,c) = (a\,b)\,c = b\,(c\,a)$, or the product of any number of factors is *independent of the order of grouping them.*

This is called the Associative Law of Multiplication.

68. If d is a positive integer, then $(a + b + c)\,d =$

$$(a + b + c) + (a + b + c) + (a + b + c) \ldots$$

repeated d times,

$$= a + b + c + a + b + c + a + b + c \ldots$$
$$= a + a + a + \ldots \text{ repeated } d \text{ times,}$$
$$+ b + b + b + \ldots \quad \text{“} \qquad \text{“}$$
$$+ c + c + c + \ldots \quad \text{“} \qquad \text{“}$$
$$= a\,d + b\,d + c\,d;$$

that is, the product of the sum of any number of algebraic numbers by a third is equal to the sum of the

products obtained by multiplying the numbers separately by the third.

This is called the Distributive Law of Multiplication.

69. The multiplier must always be an abstract number, and the product is always of the *same nature* as the multiplicand.

The cost of 5 bushels of potatoes at 75 cents a bushel is 75 cents taken, not 5 bushels times, but 5 times; and the product is of the same denomination as the multiplicand 75, viz. cents.

In Algebra the sign of the multiplier shows whether the repetitions are to be added or subtracted.

1. $$(+a) \times (+4) = +4a;$$
that is, $+a$ added 4 times is $+a+a+a+a=+4a$.

2. $$(+a) \times (-4) = -4a;$$
that is, $+a$ subtracted 4 times is $-a-a-a-a=-4a$.

3. $$(-a) \times (+4) = -4a;$$
that is, $-a$ added 4 times is $-a-a-a-a=-4a$.

4. $$(-a) \times (-4) = +4a;$$
that is, $-a$ subtracted 4 times is $+a+a+a+a=+4a$.

In the first and second examples the *nature* of the product is $+$; in the first, the $+$ sign of 4 shows that the product is to be added, and $+4a$ added is $+4a$; in the second, the $-$ sign of 4 shows that the product is to be subtracted, and $+4a$ subtracted is $-4a$. In the third and fourth examples the *nature* of the product is $-$; in the third, the $+$ sign of 4 shows that the product is to be added, and $-4a$ added is $-4a$; in the fourth, the $-$ sign of 4 shows that the product is to be subtracted, and $-4a$ subtracted is $+4a$.

70. Hence, in multiplication, we have for the sign of the product the following

Rule.

Like signs give $+$; *unlike,* $-$.

Hence the product of an *even* number of negative factors is positive; of an *odd* number, negative.

71. Multiplication in Algebra can be presented best under three cases.

Case I.

72. When both Factors are Monomials.

1. Multiply $3a$ by $2b$.

$$3a \times 2b = 3 \times a \times 2 \times b = 3 \times 2 \times a \times b = 6ab$$

As by the Commutative Law the product is the same in whatever order the factors are arranged, we have simply changed their order and united in one product the numerical coefficients.

2. Multiply a^3 by a^2.

As the exponent, if integral and positive, of a number shows how many times it is taken as a factor,

$$a^3 = a \times a \times a$$

and $$a^2 = a \times a$$

$$\therefore \; a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a \times a \times a \times a \times a = a^5$$

Therefore the product of powers of the same number is *that number with an index equal to the sum of the indices of the factors.* This is called the Index Law.

Hence, when both factors are monomials, we have the following

Rule.

Annex the product of the literal factors to the product of their coefficients, remembering that like signs give +, and unlike, −.

(3.)	(4.)	(5.)	(6.)	(7.)
$7\,x\,y$	$6\,x^2\,y^2$	$7\,a\,b$	$-24\,m\,n^3$	$-\;a^2\,b^4$
$2\,a\,b$	$3\,x\,y^3$	$-\;9\,a^2\,b$	$6\,a\,n^4$	$-9\,a^2\,b$
$14\,a\,b\,x\,y$	$18\,x^3\,y^5$	$-63\,a^3\,b^2$	$-144\,a\,m\,n^7$	$9\,a^4\,b^5$

8. $a^4 \times a^5 = ?$
9. $x^3 \times x^4 = ?$
10. $c^5 \times (-c^3) = ?$
11. $-x^4 \times x^7 = ?$
12. $-a^3 \times (-a^3) = ?$
13. $3\,a\,y^2 \times 5\,a^3\,y^4 = ?$
14. $613\,x^2\,z^3 \times 7\,x^3\,z^4 = ?$
15. $16\,x^3\,y^2 \times 5\,x^2\,y^5 = ?$
16. $-8\,c^2\,x^3 \times 5\,c^3\,x^7 = ?$
17. $-5\,a^3 y^3 \times (-4\,a^4\,y^5) = ?$
18. $14\,a^3\,x\,z \times 5\,a^2\,x^2\,z^3 = ?$
19. $-15\,x^2 y z^3 \times (-4\,x^2 y^3) = ?$
20. $304\,a^4\,b^3 \times (-8\,a^5\,b^4) = ?$
21. $706\,x^3\,y^2\,z \times 5\,x\,y^2\,z^3 = ?$
22. $16\,b^2\,c^3\,d^4 \times 2\,a\,b^3\,c^5 \times (-4\,a^3\,d^2) = ?$
23. $8\,x\,y^2\,z^4 \times (-3\,x^4\,y\,z^2) \times (-2\,x^2\,y^3\,z) = ?$
24. $15\,\sqrt{x\,y} \times 5\,a\,b \times 2\,c\,d = ?$
25. $3\,(a+b) \times 5\,(a+b)^2 = 15\,(a+b)^3.$

NOTE. Any number of terms enclosed in a bracket may be treated as a monomial.

26. $-10\,(x^3 - y^3) \times 4\,(x^3 - y^3)^3 = ?$
27. $(a-b)^4 \times (a-b)^3 \times (a-b)^2 = ?$
28. $3\,(x-z)^m \times 4\,(x-z)^n = 12\,(x-z)^{m+n}.$
29. $-8\,(c-x) \times (-3\,(c-x)^2) \times (-2\,(c-x)^3) = ?$

CASE II.

73. When only one Factor is a Monomial.

1. Multiply $x + y + z$ by a.

$$(x + y + z)\,a = a\,x + a\,y + a\,z,$$

or,

$$\begin{array}{r} x + y + z \\ a \\ \hline a\,x + a\,y + a\,z \end{array}$$

These results follow from the Distributive Law (§ 68).

Therefore, for the multiplication of a polynomial by a monomial, we have the following

Rule.

Multiply each term of the multiplicand by the multiplier, and connect the several results by their proper signs.

(2.)

$$\begin{array}{r} 2\,x^2 + 6\,x - 12\,y \\ 4\,x\,y \\ \hline 8\,x^3\,y + 24\,x^2\,y - 48\,x\,y^2 \end{array}$$

3. $(6\,a^3 - 5\,a^2\,b - 4\,a\,b^2) \times (-3\,a\,b^3) = ?$
4. $(-a\,b + b\,c - c\,a) \times (-a\,b\,c) = ?$
5. $(-5\,x\,y^2\,z + 3\,x\,y\,z^2) \times x\,y\,z = ?$
6. $(a\,b\,c - a^2\,b\,c - a\,b^2\,c) \times - a\,b\,c = ?$
7. $(-a^2\,b\,c + b^2\,c\,a - c^2\,a\,b) \times - a\,b = ?$
8. $(9\,g\,h - 12\,g\,m - 6\,g\,n) \times (-3\,g\,h) = ?$
9. $(4\,a^4\,x - 5\,a^3\,x^2 - a\,x^4 + 2\,x^5) \times (-11\,a\,x^2\,z^2) = ?$
10. $(m^3 - 3\,m^2\,n + 3\,m\,n^2 - n^3) \times n = ?$
11. $(-2\,y + 3\,z - 5\,x^2\,y^2\,z^2 - 7\,x\,z^5 + 2\,x^5\,z) \times (-3\,y\,z^3\,x) = ?$
12. $(a^2 + b^2 - c^2 + d^2 - e^2 + f^2) \times (-a^2\,b^3\,c^4) = ?$

CASE III.

74. When both Factors are Polynomials.

1. Multiply $x + y + z$ by $a + b$.

$$(x + y + z)(a + b) = (a + b)x + (a + b)y + (a + b)z,$$

by the Distributive Law, $(a + b)$ being regarded as a single term. The last expression further reduced by the same law becomes $ax + bx + ay + by + az + bz$, and this equals by the Commutative Law for Addition (§ 56), $ax + ay + az + bx + by + bz$.

Hence, for the multiplication of a polynomial by a polynomial, we have the following

Rule.

Multiply each term of the multiplicand by each term of the multiplier, and find the sum of the several products.

2. Multiply $2x^2 + 3xy - y^2$ by $3x - 2y$.

$$\begin{array}{l} 2x^2 + 3xy - y^2 \\ 3x - 2y \\ \hline 6x^3 + 9x^2y - 3xy^2 \\ \qquad - 4x^2y - 6xy^2 + 2y^3 \\ \hline 6x^3 + 5x^2y - 9xy^2 + 2y^3 \end{array}$$

We begin at the left, placing the second result one place to the right, so that like terms may stand in the same vertical column.

3. Multiply $3x - 3x^2 - 1 + x^3$ by $3x + x^2 + 1$.

$$\begin{array}{lllllll} x^3 & - 3x^2 & + 3x & - 1 & & & \\ x^2 & + 3x & + 1 & & & & \\ \hline x^5 & - 3x^4 & + 3x^3 & - x^2 & & & \\ & + 3x^4 & - 9x^3 & + 9x^2 & - 3x & & \\ & & + x^3 & - 3x^2 & + 3x & - 1 & \\ \hline x^5 & & - 5x^3 & + 5x^2 & & - 1 & \end{array}$$

In Examples 2 and 3 the multiplicand and multiplier were arranged according to the descending powers of x before multiplying. The polynomials could have been arranged according to the ascending powers as well.

NOTE. Though the arrangement of the polynomials according to the ascending or descending powers of some letter is not absolutely essential, it should be observed where possible, as symmetry of work tends to minimize errors.

4. Multiply $2xz - z^2 + 2x^2 - 3yz + xy$ by $x - y + 2z$.

$$
\begin{array}{lllllllll}
2x^2 & + xy & + 2xz & - 3yz & - z^2 & & & & \\
x & - y & + 2z & & & & & & \\
\hline
2x^3 & + x^2y & + 2x^2z & - 3xyz & - xz^2 & & & & \\
 & - 2x^2y & & - 2xyz & & - xy^2 & + 3y^2z & + yz^2 & \\
 & & + 4x^2z & + 2xyz & + 4xz^2 & & & - 6yz^2 & - 2z^3 \\
\hline
2x^3 & - x^2y & + 6x^2z & - 3xyz & + 3xz^2 & - xy^2 & + 3y^2z & - 5yz^2 & - 2z^3
\end{array}
$$

5. $(2x^2 + 3xy - y^2) \times (3x - 2y) = ?$

6. $(3x^4 - x^2 - 1) \times (2x^4 - 3x^2 + 7) = ?$

7. $(b^4 - 2b^2 + 1) \times (b^4 + 2b^2 + 1) = ?$

8. $(x^2 + ax + a^2) \times (x - a) = ?$

9. $(x^4 - ax^3 + a^2x^2 - a^3x + a^4) \times (x + a) = ?$

10. $(x^3 - x^2y + xy^2 - y^3) \times (x^2 + 2xy + y^2) = ?$

11. $(a^3 + 2a^2b + 4ab^2 + 8b^3) \times (a^2 - 4ab + 4b) = ?$

12. $(a^2 + b^2 + c^2 + ab + ac - bc) \times (a - b - c) = ?$

13. $(4a^2 + 9b^2 + c^2 - 3bc - 2ac - 6ab) \times (2a + 3b + c) = ?$

Ans. $8a^3 - 18abc + 27b^3 + c^3$.

14. $(a + b - c)(b + c - a)(c + a - b) \times (a + b + c) = ?$

15. $(ab + cd + ac + bd) \times (ab + cd - ac - bd) = ?$

16. $(-3a^2b^2 + 4ab^3 + 15a^3b) \times (5a^2b^2 + ab^3 - 3b^4) = ?$

17. $(48a^2x - 64a^3 + 27x^3 - 36ax^2) \times (3x + 4a) = ?$

Ans. $81x^4 - 256a^4$.

18. $(a^2 - 5ab - b^2) \times (a^2 + 5ab + b^2) = ?$

19. $(x^2 - x\,y + x + y^2 + y + 1) \times (x + y - 1) = ?$

20. $(a^2 + b^2 + c^2 - b\,c - a\,c - a\,b) \times (a + b + c) = ?$

21. $(x^{12} - x^9\,y^2 + x^6\,y^4 - x^3\,y^6 + y^8) \times (x^3 + y^2) = ?$

22. $(2\,a^3 + 2\,a + 1 + a^4 + 3\,a^2) \times (1 - 2\,a + a^2) = ?$

23. $(3\,a\,x\,y^2 - 9\,a\,y^4 - a\,x^2) \times (-\,a\,x - 3\,a\,y^2) = ?$

24. $(3\,x^2y^2 - 2\,x\,y^3 - 2\,x^3y + x^4 + y^4) \times (x^2 + 2\,x\,y + y^2) = ?$

25. $(x^5 + 4\,x^4\,y^2 + 8\,x^3\,y^4 + 15\,x^2\,y^6) \times (-\,x^3 + 2\,x^2y^2 - y^6) = ?$

Ans. $-\,x^8 - 2\,x^7\,y^2 + 26\,x^4\,y^8 - 8\,x^3\,y^{10} - 15\,x^2\,y^{12}$.

26. $(x^4 + x^3\,y + x^2\,y^2 + x\,y^3) \times (-\,x^4\,y^2 + x\,y^5 + x^5\,y - y^6) = ?$

27. $(x^3 - y^3 + 3\,x\,y^2 + 2\,x^2\,y) \times (2\,x\,y - x^2 - y^2) = ?$

28. $(21\,x^2\,y - 14\,x\,y^2 - 7\,y^3) \times (-\,3\,x^4 - x^2\,y^2 + y^4) = ?$

29. $(2\,a^3 + 3\,a^2\,b + 3\,a\,b^2 + 2\,b^3) \times (4\,a^4\,b - 6\,a^3\,b^2 + 6\,a^2\,b^3 - 4\,a\,b^4) = ?$

Ans. $8\,a^7b + 6\,a^5b^3 - 6\,a^3b^5 - 8\,a\,b^7$.

When the coefficients are fractional, the ordinary process is still employed.

30. Multiply $\frac{1}{3}\,a^2 - \frac{1}{2}\,a\,b + \frac{2}{3}\,b^2$ by $\frac{1}{2}\,a + \frac{1}{3}\,b$.

$$
\begin{array}{llll}
\frac{1}{3}\,a^2 & - \frac{1}{2}\,a\,b & + \frac{2}{3}\,b^2 & \\
\frac{1}{2}\,a & + \frac{1}{3}\,b & & \\
\hline
\frac{1}{6}\,a^3 & - \frac{1}{4}\,a^2\,b & + \frac{1}{3}\,a\,b^2 & \\
 & + \frac{1}{9}\,a^2\,b & - \frac{1}{6}\,a\,b^2 & + \frac{2}{9}\,b^3 \\
\hline
\frac{1}{6}\,a^3 & - \frac{5}{36}\,a^2\,b & + \frac{1}{6}\,a\,b^2 & + \frac{2}{9}\,b^3
\end{array}
$$

31. $(\frac{1}{2}\,a^2 + \frac{1}{3}\,a + \frac{1}{4}) \times (\frac{1}{2}\,a - \frac{1}{3}) = ?$

32. $(\frac{1}{2}\,x^2 - 2\,x + \frac{3}{2}) \times (\frac{1}{2}\,x + \frac{1}{3}) = ?$

33. $(\frac{2}{3}\,x^2 + x\,y + \frac{3}{2}\,y^2) \times (\frac{1}{3}\,x - \frac{1}{2}\,y) = ?$

34. $(\frac{3}{2}\,x^2 - a\,x - \frac{2}{3}\,a^2) \times (\frac{3}{4}\,x^2 - \frac{1}{2}\,a\,x + \frac{1}{3}\,a^2) = ?$

Ans. $\frac{9}{8}\,x^4 - \frac{3}{2}\,a\,x^3 + \frac{1}{2}\,a^2\,x^2 - \frac{2}{9}\,a^4$.

75. A somewhat abridged and simple method of working examples in which the exponents of the letters increase and decrease by a common difference, as in Example 3, is to omit the letters altogether. Thus:

(1.)

$$\begin{array}{l}
1-3+3-1 \\
1+3+1 \\
\hline
1-3+3-1 \\
\quad +3-9+9-3 \\
\qquad +1-3+3-1 \\
\hline
1+0-5+5+0-1
\end{array}$$

Ans. $x^5 + 0\,x^4 - 5\,x^3 + 5\,x^2 + 0\,x - 1$,
or, $x^5 - 5\,x^3 + 5\,x^2 - 1$.

The insertion of the powers of x depends upon the fact that the highest power in the product is always the product of the highest powers in the two factors, and that the rest follow in order.

2. Multiply $x^3 + 5x - 3$ by $x^2 - 1$, or $x^3 + 0x^2 + 5x - 3$ by $x^2 + 0\,x - 1$.

$$\begin{array}{l}
1+0+5-3 \\
1+0-1 \\
\hline
1+0+5-3 \\
\qquad -1-0-5+3 \\
\hline
1+0+4-3-5+3
\end{array}$$

Ans. $x^5 + 0\,x^4 + 4\,x^3 - 3\,x^2 - 5\,x + 3$,
or, $x^5 + 4\,x^3 - 3\,x^2 - 5\,x + 3$.

The insertion of zero coefficients at the beginning of the above operation, as well as at the close, is necessary in order to preserve the law of the exponents.

3. Multiply $a^3 - 3a^2b + 3ab^2 + b^3$ by $a^2 - 2ab + b^2$.

4. Multiply $5x^3 - 3ax^2 + 5a^2x - a^3$ by $a^2 + 3ax + 5x^2$.

5. Multiply $4x^2 - 24xy + 36y^2$ by itself.

6. Multiply $x^3 - xy^2 - y^3$ by $x^3 + 3y^3$.

The process above is called Multiplication by Detached Coefficients. Many of the examples in § 74 will serve as additional exercises under this method.

76. Exercises in the Omission and Insertion of Brackets.

A coefficient placed before a bracket indicates that every term of the expression within the bracket is to be multiplied by that coefficient.

1. Simplify $94 - 8[-11x - 4\{-17x + 3(8 - \overline{9 - 5x})\}]$.

$$\begin{aligned}
&94 - 8[-11x - 4\{-17x + 3(8 - 9 + 5x)\}] \\
&= 94 - 8[-11x - 4\{-17x + 15x - 3\}] \\
&= 94 - 8[-11x - 4\{-2x - 3\}] \\
&= 94 - 8[-11x + 8x + 12] \\
&= 94 - 8[-3x + 12] \\
&= 94 + 24x - 96 \\
&= 24x - 2
\end{aligned}$$

This example can be done much more briefly as follows:

$$\begin{aligned}
&94 - 8[-11x - 4\{-17x + 3(8 - \overline{9 - 5x})\}] \\
&= 94 + 88x + 32\{-17x + 24 - 27 + 15x\} \\
&= 94 + 88x + 32\{-2x - 3\} \\
&= 94 + 88x - 64x - 96 \\
&= 24x - 2.
\end{aligned}$$

Simplify:

2. $b - (c - a) - [b - a - c - 2\{c + a - 3(a - b) - d\}]$.

3. $-20(a - d) + 3(b - c) - 2[b + c + d - 3\{c + d - 4(d - a)\}]$. Ans. $4a + b + c$.

4. $a - 2(b - c) - [-\{-(4a - b - c - 2\{a + b + c\})\}]$.

5. $-3\{-2[-4(-a)]\} + 5\{-2[-2(-a)]\}$.

6. $x^2 - [(x - y)^2 - \{(x - y - z)^2 - (z - x)^2\}]$.

7. $x^2 - [x^2 + y^2 + (x - y)(x + y) - \{x^2 - (y^2 - \overline{x^2 + y^2})\}]$.

Ans. x^2.

8. $(2a - b)(2a + b) + [ab - b\{a - (2a - \overline{2a - b})\}]$.

9. $(a + b + c)^2 - (a + b)(a - c) - (a - b)(b - c) - (b - c)^2 - b^2$.

10. $(a + b + c)^2 - a(b + c - a) - b(a + c - b) - c(a + b - c)$.

Ans. $2(a^2 + b^2 + c^2)$.

In the following expressions bracket together the equal powers of x, so that the signs before all the brackets shall be positive.

11. $ax^4 + bx^2 + 2bx - 5x^2 + 2x^4 - 3x$.

$$ax^4 + bx^2 + 2bx - 5x^2 + 2x^4 - 3x$$
$$= ax^4 + 2x^4 + bx^2 - 5x^2 + 2bx - 3x$$
$$= (a + 2)x^4 + (b - 5)x^2 + (2b - 3)x.$$

12. $3bx^2 - 2x + 5ax^3 + cx - 4x^2 - bx^3$.

13. $-7x^3 + 5ax^2 - 2cx + 9ax^3 + 7x - 3x^2$.

14. $2cx^5 - 3abx + 4dx - 3bx^4 - a^2x^5 + x^4$.

In the following expressions bracket together the equal powers of x, so that the signs before all the brackets shall be negative.

15. $ax^2 + 5x^3 - a^2x^4 - 2bx^3 - 3x^2 - 4x^4$.

16. $7x^3 - 3c^2x - abx^5 + 5ax + 7x^5 - abcx^3$.

17. $ax^2 + a^2x^3 - bx^2 - 5x^2 - cx^3$.

18. $3b^2x^4 - bx - ax^4 - cx^4 - 5c^2x - 7x^4$.

Simplify the following expressions, and in each result group the terms according to the powers of x.

19. $a x^3 - 2 c x - [b x^2 - \{c x - d x - (b x^3 + 3 c x^2)\} - (cx^2 - b x)]$.

20. $5 a x^3 - 7(b x - c x^2) - \{6 b x^2 - (3 a x^2 + 2 a x) - 4 c x^3\}$.

21. $x \{x - b - x (a - b x)\} + \{a x - x (a x - b)\}$.

Add together the following expressions in each example, and group the results according to the powers of x.

22. $a x^3 - 2 b x^2$, $b x - c x^3 - x^2$, and $x^3 - a x^2 + c x$.

23. $a^2 x^3 - 5 x$, $2 a x^2 - 5 a x^3$, and $2 x^3 - b x^2 - a x$.

24. $a x^2 + b x - c$, $q x - r - p x^2$, and $x^2 + 2 x + 3$.

Multiply together the following expressions in each example, and group the results according to the powers of x.

25. $a x^2 - 2 b x + 3 c$ and $p x - q$.

$$\begin{aligned} & (a x^2 - 2 b x + 3 c)\ (p x - q) \\ = {} & a p x^3 - 2 b p x^2 + 3 c p x - a q x^2 + 2 b q x - 3 c q \\ = {} & (a p)\, x^3 - (2 b p + a q)\, x^2 + (3 c p + 2 b q)\, x - (3 c q) \end{aligned}$$

26. $(x^3 + a x^2 - b x - c)\ (x^3 - a x^2 - b x + c)$.

27. $(x^4 - a x^3 - b x^2 + c x)\ (x^4 + a x^3 - b x^2 - c x + d)$.

28. $(x^2 - a x - b x^2)\ (x^2 + p x - q^2)$.

29. $(x^2 - 3 a x + 6 a^2)\ (x^2 + 5 b x + 8 b^2)$.

30. $(x^3 - 5 b x^2 + 3 b)\ (x^4 + 3 b x^3 - 2 b x + b)$.

31. $(x^4 - 3 d x^2 + d)\ (x^3 + 2 c x + c)$.

32. $(x^5 + 5 m x^3 - 3 n x)\ (x^4 - 4 m x^2 + 2 n)$.

33. $(x^6 - 4 a x^4 + 2 b x^2 + c)\ (x^4 + 2 a x^2 - b)$.

34. $(x^7 + 3 m x^4 - 2 n x)\ (x^3 - 2 m x + n)$.

CHAPTER V.

DIVISION.

77. Division is finding a quotient which, multiplied by the divisor, will produce the dividend. Division is the inverse of multiplication.

In accordance with this definition and the Rule in § 70, the sign of the quotient must be $+$ when the divisor and the dividend have like signs; $-$ when the divisor and the dividend have unlike signs; that is, in division, as in multiplication, we have for the signs the following

Rule.

Like signs give $+$; *unlike,* $-$.

Case I.

78. When the Divisor and Dividend are both Monomials.

1. Divide $9ax$ by $3x$.

$$9ax \div 3x = 3a$$

The coefficient of the quotient must be a number which, multiplied by 3, the coefficient of the divisor, will give 9, the coefficient of the dividend, that is, 3; and the literal part of the quotient must be a number which, multiplied by x, will give ax, that is, a; the quotient required, therefore, is $3a$.

2. Divide a^5 by a^2.

$$a^5 \div a^2 = a^3, \text{ or } \frac{a\,a\,a\,a\,a}{a\,a} = a\,a\,a = a^3$$

For (§ 72), $a^3 \times a^2 = a\,a\,a \times a\,a = a^5$. Therefore the quotient of two powers of the same number is *that number with an index equal to the index of the dividend minus the index of the divisor.*

Hence, for the division of monomials we have the following

Rule.

Annex the quotient of the letters to the quotient of their coefficients, remembering that like signs give + and unlike —.

(3.)

$$\frac{36\,a^2\,x^3}{9\,a\,x^2} = 4\,a\,x$$

(4.)

$$\frac{42\,b^4\,x^3\,y}{7\,b^2\,x\,y} = 6\,b^2\,x^2$$

(5.)

$$\frac{-412\,a^4\,y^5}{103\,a^3\,y^2} = -4\,a\,y^3$$

(6.)

$$\frac{-54\,c^5\,x^2\,y^4}{-6\,c^3\,x^2\,y} = 9\,c^2\,y^3$$

7. $26\,x^2\,y^4\,z \div 2\,x^2\,y^3 = ?$

8. $475\,a^2\,b^3\,c^4 \div 25\,a\,b^2\,c^3 = ?$

9. $-85\,x^3\,y^2\,z \div 5\,x\,y^2 = ?$

10. $68\,b^5\,c^3\,d^2 \div (-17\,b^2\,c^2\,d^5) = ?$

11. $-135\,a^2\,m^4\,n^3 \div (-15\,a\,m^4) = ?$ Ans. $9\,a\,n^3$.

12. $b^5\,z^3 \div b^2\,z = ?$

13. $-16\,a^7 \div 4\,a^5 = ?$

14. $128\,a^5\,x^4 \div (-16\,a^3\,x) = ?$

15. $-310\,c^5\,m^2 \div (-10\,c^3\,m) = ?$

16. $238\,a^5\,y^3 \div (-7\,a^4\,y^3) = ?$

17. $a^m \div a^n = a^{m-n}$.

18. $27\,x^m\,y^n \div 9\,x^n\,y^3 = ?$ Ans. $3\,x^{m-n}\,y^{n-3}$.

19. $342\,x\,y^3\,z^5 \div (-114\,x\,y^2\,z^5) = ?$

20. $135\,(a+b)^5 \div 9\,(a+b)^3 = ?$ Ans. $15\,(a+b)^2$.

21. $27\,(a-b)^4 \div 3\,(a-b)^3 = ?$

22. $-34\,(m-n)^6 \div 17\,(m-n)^3 = ?$

23. $123\,(c-d)^5 \div (-41\,\{c-d\}^3) = ?$

79. In this book when successive numbers are separated by the sign $\times$ or $\div$, the operations thus indicated are to be completed in succession *from left to right.* Thus,

$a \div b \times c = \frac{a}{b} \times c = \frac{ac}{b}$, and is not equal to $a \div (b \times c)$, or $\frac{a}{bc}$.

So $24 \div 4 \times 3 = \frac{24}{4} \times 3 = 18$, and is not equal to $\frac{24}{4 \cdot 3}$, or 2.

From this it will be seen that

$$a \div b \times c \quad = a \div (b \div c)$$

and

$$a \div (b \times c) = a \div b \div c$$

That is, as in addition and subtraction, the removal of a bracket with the sign $+$ before it requires no change of signs, while the removal of a bracket with the sign $-$ before it requires that the sign of each term taken out should be changed from $+$ to $-$, or from $-$ to $+$ (§ 62), so, in multiplication and division, the removal of a bracket with the sign $\times$ before it requires no change of signs, while the removal of a bracket with the sign $\div$ before it requires that the signs of the successive numbers separated by the sign $\times$ or $\div$ should be changed from $\times$ to $\div$, or from $\div$ to $\times$. Thus,

$$7 \times (a \times b \div c \times d) = 7 \times a \times b \div c \times d = \frac{7\,abd}{c}$$

but $7 \div (a \times b \div c \times d) = 7 \div a \div b \times c \div d = \frac{7\,c}{abd}$

So $5\,abc \div d = 5 \times (a \times b \times c \div d) = 5 \div (d \div a \div b \div c)$

1. Divide $y^8 \times y^4 \div y^2 \times y$ by $y^8 \div y^4 \times y^2 \div y$.

$$\begin{aligned}
&(y^8 \times y^4 \div y^2 \times y) \div (y^8 \div y^4 \times y^2 \div y) \\
&= y^8 \times y^4 \div y^2 \times y \div y^8 \times y^4 \div y^2 \times y \\
&= y^8 \times y^4 \times y \times y^4 \times y \div y^2 \div y^8 \div y^2 \\
&= (y^8 \times y^4 \times y \times y^4 \times y) \div (y^2 \times y^8 \times y^2) \\
&= y^{18} \div y^{12} = y^6
\end{aligned}$$

The process above is written out merely to illustrate the changes of signs involved in removing and introducing a bracket with the sign $\div$ before it, and the principle that must be used in applying the Commutative Law in multiplication and division.

2. $(2a \div a \times a) \div (2a \times a \div a) = ?$ Ans. 1.

3. $(24x^2 \div 2x \div 3) \div (2x \div 2) = ?$

4. $(5 \div 2 \times 10) \div (10 \div 5 \times 2) = ?$ Ans. $6\frac{1}{4}$.

5. $(32 \div 8 \times 4 \div 2) \div (12 \div 6 \times 2) = ?$

CASE II.

80. When the Divisor only is a Monomial.

1. Divide $ax + ay + az$ by a.

From § 73, $(x + y + z)\,a = ax + ay + az$. Conversely, $(ax + ay + az) \div a = x + y + z$, so that the quotient obtained by dividing the sum of two or more monomials by a third is the sum of the quotients obtained by dividing the monomials separately by the third. Hence the following

Rule.

Divide each term of the dividend by the divisor, and connect the several results by their proper signs.

(2.)

$$4a)\,\underline{28ax^3 - 12ax^4}$$
$$7x^3 \quad - \quad 3x^4$$

(3.)

$$-3x^3y^2)\,\underline{-21x^4y^2 - 18x^5y^2}$$
$$7x \quad + \quad 6x^2$$

4. $(a^2 - ab - ac) \div (-a) = ?$

5. $(16x^6 - 24x^8 - 32x^4) \div (-8x^3) = ?$

6. $(3x^3 - 9x^2y - 12xy^2) \div 3x = ?$

7. $(4x^4y^4 - 8x^3y^2 + 6xy^3) \div (-2xy) = ?$

8. $(10a^4x^2 - 30a^3x^3 + 40a^2x^2y^2) \div 5a^2x = ?$

9. $(-8a^4x^2 + 16a^2b^2x^2 - 12b^4x^3) \div 4x^2 = ?$

10. $(9x - 12y + 3y) \div (-3) = ?$

11. $(36a^3b^2 - 24a^2b^5 - 20a^4b^2) \div 4a^2b = ?$

12. $(2x^2 - 5xy + \frac{3}{2}x^2y^3) \div (-\frac{1}{2}x) = ?$

13. $(-3a^2 - \frac{9}{2}ab - 6ac) \div (-\frac{3}{2}a) = ?$

14. $(\frac{1}{4}a^2x - \frac{1}{16}abc - \frac{3}{8}acx) \div \frac{3}{8}ax = ?$

Case III.

81. When the Divisor and Dividend are both Polynomials.

1. Divide $a^3 - 3\,a^2\,b + 3\,a\,b^2 - b^3$ by $a^2 - 2\,a\,b + b^2$.

$$\begin{array}{r|l} a^2 - 2\,a\,b + b^2) & a^3 - 3\,a^2\,b + 3\,a\,b^2 - b^3\,(a - b \\ & \underline{a^3 - 2\,a^2\,b + \,a\,b^2} \\ & -\,a^2\,b + 2\,a\,b^2 - b^3 \\ & \underline{-\,a^2\,b + 2\,a\,b^2 - b^3} \end{array}$$

The divisor and dividend are arranged in the order of the powers of a, beginning with the highest power. a^3, the highest power of a in the dividend, must be the product of the highest power of x in the quotient and a^2 in the divisor; therefore, $\frac{a^3}{a^2} = a$ must be the highest power of a in the quotient. The divisor, $a^2 - 2\,a\,b + b^2$, multiplied by a, must give several of the partial products which would be produced were the divisor multiplied by the whole quotient. When $(a^2 - 2\,a\,b + b^2)\,a = a^3 - 2\,a^2\,b + a\,b^2$ is subtracted from the dividend, the remainder must be the product of the divisor and the remaining terms of the quotient; therefore we treat the remainder as a new dividend, and so continue until the dividend is exhausted.

Hence, for the division of polynomials, we have the following

Rule.

Arrange the divisor and dividend in the order of the powers of one of the letters.

Divide the first term of the dividend by the first term of the divisor; the result will be the first term of the quotient.

Multiply the whole divisor by this quotient, and subtract the product from the dividend.

Consider the remainder as a new dividend, and proceed as before until the dividend is exhausted.

2. Divide $8a^3 + 8a^2b + 4ab^2 + b^3$ by $2a + b$.

$$
\begin{array}{l}
2a + b)\, 8a^3 + 8a^2b + 4ab^2 + b^3\, (4a^2 + 2ab + b \\
\qquad\quad \underline{8a^3 + 4a^2b} \\
\qquad\qquad\qquad 4a^2b + 4ab^2 \\
\qquad\qquad\qquad \underline{4a^2b + 2ab^2} \\
\qquad\qquad\qquad\qquad\quad 2ab^2 + b^3 \\
\qquad\qquad\qquad\qquad\quad \underline{2ab^2 + b^3}
\end{array}
$$

3. Divide $a^4 - a^3b + 2a^2b^2 - ab^3 + b^4$ by $a^2 + b^2$.

$$
\begin{array}{l}
a^2 + b^2)\, a^4 - a^3b + 2a^2b^2 - ab^3 + b^4\, (a^2 - ab + b^2 \\
\qquad\quad \underline{a^4 \qquad\quad + \;\; a^2b^2} \\
\qquad\qquad - a^3b + \;\; a^2b^2 - ab^3 \\
\qquad\qquad \underline{- a^3b \qquad\qquad\quad - ab^3} \\
\qquad\qquad\qquad\quad a^2b^2 \qquad\quad + b^4 \\
\qquad\qquad\qquad\quad \underline{a^2b^2 \qquad\quad + b^4}
\end{array}
$$

4. Divide $x^4 - 1$ by $x - 1$.

$$
\begin{array}{l}
x - 1)\, x^4 + 0x^3 + 0x^2 + 0x - 1\, (x^3 + x^2 + x + 1 \\
\qquad\quad \underline{x^4 - \;\; x^3} \\
\qquad\qquad + \;\; x^3 + 0x^2 \\
\qquad\qquad\quad\; \underline{x^3 - \;\; x^2} \\
\qquad\qquad\qquad\; + \;\; x^2 + 0x \\
\qquad\qquad\qquad\; \underline{+ \;\; x^2 - \;\; x} \\
\qquad\qquad\qquad\qquad\qquad x - 1 \\
\qquad\qquad\qquad\qquad\qquad \underline{x - 1}
\end{array}
$$

or,

$$
\begin{array}{l}
x - 1)\, x^4 - 1\, (x^3 + x^2 + x + 1 \\
\qquad\qquad \underline{x^4 - x^3} \\
\qquad\qquad\qquad x^3 - 1 \\
\qquad\qquad\qquad \underline{x^3 - x^2} \\
\qquad\qquad\qquad\qquad x^2 - 1 \\
\qquad\qquad\qquad\qquad \underline{x^2 - x} \\
\qquad\qquad\qquad\qquad\qquad x - 1 \\
\qquad\qquad\qquad\qquad\qquad \underline{x - 1}
\end{array}
$$

5. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

$$
\begin{array}{l|l}
a^3 - 3abc + b^3 + c^3 & a + b + c \\
\cline{2-2}
\underline{a^3 + a^2 b + a^2 c} & a^2 - ab - ac + b^2 - bc + c^2. \\
\quad - a^2 b - a^2 c - 3abc & \\
\quad \underline{- a^2 b - ab^2 - abc} & \\
\qquad - a^2 c + ab^2 - 2abc & \\
\qquad \underline{- a^2 c - abc - ac^2} & \\
\qquad\qquad ab^2 - abc + ac^2 + b^3 & \\
\qquad\qquad \underline{ab^2 + b^3 + b^2 c} & \\
\qquad\qquad\quad - abc + ac^2 - b^2 c & \\
\qquad\qquad\quad \underline{- abc - b^2 c - bc^2} & \\
\qquad\qquad\qquad ac^2 + bc^2 + c^3 & \\
\qquad\qquad\qquad \underline{ac^2 + bc^2 + c^3} &
\end{array}
$$

In the examples the dividend, divisor, and successive remainders are arranged in *descending* powers of a and x. The *ascending* powers would have answered as well. The choice of letter and kind of arrangement are immaterial, but it is especially important before beginning the division that some arrangement should be adopted and maintained throughout the operation.

6. $(x^2 + 12x + 20) \div (x + 2) = ?$

7. $(x^2 - 3x - 70) \div (x + 7) = ?$

8. $(2a^3 - 7a^2 - 3a + 18) \div (2a + 3) = ?$

9. $(x^3 + ax^2 - 3a^2 x - 6a^3) \div (x - 2a) = ?$

10. $(x^4 + 4x^2 + 16) \div (x^2 + 2x + 4) = ?$

11. $(x^5 - 1) \div (x - 1) = ?$

12. $(x^{12} + y^{12}) \div (x^4 + y^4) = ?$

13. $(x^3 - 8y^3) \div (x - 2y) = ?$

14. $(x^5 + y^5) \div (x + y) = ?$

15. $(x^5 - 32y^5) \div (x - 2y) = ?$

16. $(x^4 - 8x^3 + 21x^2 - 16x - 7) \div (x^2 - 5x + 7) = ?$

17. $(4x - 2x^3 + 2x^4 + 5 - 5x^2) \div (2x + 1 + x^2) = ?$

18. $(2x^2 + x + 2 + x^4) \div (x + 1 + x^2) = ?$

19. $(17x^3 + 7x + x^5 - 10x^4 + 41x^2 - 6) \div (-5x + x^2 - 6) = ?$

20. $(9x^4 - 9ax^3 - 13a^2x^2 - 16a^3x - 6a^4) \div (3x^2 - 5ax - 3a^2) = ?$

21. $(14a^4 - 45a^3b + 16a^2b^2 - 15ab^3 - 18b^4) \div (7a^2 - 5ab + 6b^2) = ?$

22. $(5xy^3 - 10y^4 + 16x^4 - 46x^3y - 21x^2y^2) \div (2y^2 + 8x^2 - 3xy) = ?$ Ans. $2x^2 - 5xy - 5y^2$.

23. $(9xy^4 - 2y^5 + 11x^3y^2 + x^5 - 5x^4y - 14x^2y^3) \div (2y^2 + x^2 - 3xy) = ?$

24. $(a^5 + 243b^5) \div (a + 3b) = ?$

25. $(x^8 - y^8) \div (x^3 + x^2y + xy^2 + y^3) = ?$

26. $(x^4 + 8x^3 - 4x^2 - 128x - 192) \div (x^2 - 16) = ?$

27. $(a^{12} + 2a^6b^6 + b^{12}) \div (a^4 + 2a^2b^2 + b^4) = ?$

28. $(x^5 - x^4y + x^3y^2 - x^3 - y^3) \div (x^3 - x - y) = ?$

29. $(3x^3 + 3x^2 + x^5 - 4x^4 - 3x + 2) \div (-x + x^2 - 2) = ?$

30. $(9x^3 + x^6 + 2 - x - 6x^2 - 5x^4) \div (2 + x^2 - 3x) = ?$

31. $(14x^4 + 78x^2y^2 + 45x^3y + 45xy^3 + 14y^4) \div (2x^2 + 5xy + 7y^2) = ?$ Ans. $7x^2 + 5xy + 2y^2$.

32. $(3x^4 + x + 2x^2 + 1 - x^3) \div (1 + x^2 - x) = ?$

33. $(2x^6 + 1 - 3x^4) \div (1 + 2x + x^2) = ?$

34. $(5x^2y^3 + y^5 - x^5 - 5xy^4) \div (2xy - x^2 - y^2) = ?$

35. $(26x^4y^8 - 8x^3y^{10} - x^8 - 2x^7y^2 - 15x^2y^{12}) \div (2x^2y^2 - x^3 - y^6) = ?$ Ans. $x^5 + 4x^4y^2 + 8x^3y^4 + 15x^2y^6$.

36. $(1 - 3ab - 29a^3b^3 + 21a^4b^4) \div (-1 + 5ab - 3a^2b^2) = ?$

37. $(14x^3y^4 + 28x^2y^5 - 63x^6y + 42x^5y^2 - 14xy^6 - 7y^7) \div (-3x^4 - x^2y^2 + y^4) = ?$

38. $(x^6 - y^6) \div (x^3 + 2x^2y + 2xy^2 + y^3) = ?$

39. $(x^8 + x^4 y^4 + y^8) \div (x^2 \div x y + y^2) = ?$

40. $(x^6 - x^4 y^2 + 3 x^3 y^3 - x^2 y^4 + y^6) \div (x^2 - x y + y^2) = ?$

41. $(a^5 - a^3 y^2 + y^5 - a^2 y^3) \div (2 a^2 y + y^3 + a^3 + 2 a y^2) = ?$

42. $(x^9 y - x y^9) \div (x^4 + x^3 y + x^2 y^2 + x y^3) = ?$

43. $(a^3 + 3 a b c + b^3 - c^3) \div (a + b - c) = ?$

44. $(a^3 + 3 a^2 b + 3 a b^2 + b^3 - 1) \div (a + b - 1) = ?$

45. $(a^3 + 6 a b c - 8 b^3 + c^3) \div (a - 2 b + c) = ?$

46. $(a^4 + b^4 + c^4 - 2 a^2 b^2 - 2 a^2 c^2 - 2 b^2 c^2) \div (a^2 + 2 a b + b^2 - c^2) = ?$ Ans. $a^2 - 2 a b + b^2 - c^2$.

47. $(a^2 b^2 + c^2 d^2 - a^2 c^2 - b^2 d^2) \div (a b + c d + a c + b d) = ?$

48. $(2 a^{3n} - 6 a^{2n} b^n + 6 a^n b^{2n} - 2 b^{3n}) \div (a^n - b^n) = ?$

49. $(6 x^{m+3n} - 19 x^{m+2n} + 20 x^{m+n} - 7 x^m - 4 x^{m-n}) \div (3 x^{2n} - 5 x^n + 4) = ?$ Ans. $2 x^{m+n} - 3 x^m - x^{m-n}$.

50. $(a^{2n} - b^{4n} + a^n c^{3n} + b^{2n} c^{3n} - a^n d^{4n} - b^{2n} d^{4n}) \div (a^n + b^{2n}) = ?$

When the coefficients are fractional, the ordinary process may still be employed.

51. Divide $\frac{1}{4} x^3 + \frac{19}{150} x y^2 + \frac{1}{15} y^3$ by $\frac{1}{2} x + \frac{1}{5} y$.

$$
\begin{array}{l}
\tfrac{1}{2} x + \tfrac{1}{5} y)\ \tfrac{1}{4} x^3 + \tfrac{19}{150} x y^2 + \tfrac{1}{15} y^3\ (\tfrac{1}{2} x^2 - \tfrac{1}{5} x y + \tfrac{1}{3} y^2 \\
\qquad\qquad \underline{\tfrac{1}{4} x^3 + \tfrac{1}{10} x^2 y \qquad\qquad\qquad\qquad} \\
\qquad\qquad\qquad - \tfrac{1}{10} x^2 y + \tfrac{19}{150} x y^2 + \tfrac{1}{15} y^3 \\
\qquad\qquad\qquad \underline{- \tfrac{1}{10} x^2 y - \tfrac{1}{25} x y^2 \qquad\qquad} \\
\qquad\qquad\qquad\qquad\qquad\quad \tfrac{1}{6} x y^2 + \tfrac{1}{15} y^3 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{\tfrac{1}{6} x y^2 + \tfrac{1}{15} y^3}
\end{array}
$$

52. $(\frac{1}{27} a^3 - \frac{1}{12} a^2 + \frac{1}{16} a - \frac{1}{64}) \div (\frac{1}{3} a - \frac{1}{4}) = ?$

53. $(\frac{1}{4} x^3 + \frac{1}{72} x y^2 + \frac{1}{12} y^3) \div (\frac{1}{2} x + \frac{1}{3} y) = ?$

54. $(\frac{9}{16}a^4 - \frac{3}{4}a^3 - \frac{7}{4}a^2 + \frac{4}{3}a + \frac{16}{9}) \div (\frac{3}{2}a^2 - \frac{8}{3} - a) = ?$

55. $(36x^2 + \frac{1}{9}y^2 + \frac{1}{4} - 4xy - 6x + \frac{1}{3}y) \div (6x - \frac{1}{3}y - \frac{1}{2}) = ?$

82. When the exponents of the letters in the divisor and dividend increase or decrease by a common difference, the abridged and simple method made use of in Multiplication, § 75, can be employed with equal facility in Division.

1. Divide $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ by $a^2 - 2ax + x^2$.

$$\begin{array}{llllll|l}
1 & -5 & +10 & -10 & +5 & -1 & 1-2+1 \\
\cline{7-7}
1 & -2 & +1 & & & & 1-3+3-1 \\
\cline{1-3}
 & -3 & +9 & -10 & & & \\
 & -3 & +6 & -3 & & & \\
\cline{2-4}
 & & 3 & -7 & +5 & & \\
 & & 3 & -6 & +3 & & \\
\cline{3-5}
 & & & -1 & +2 & -1 & \\
 & & & -1 & +2 & -1 & \\
\cline{4-6}
\end{array}$$

Ans. $a^3 - 3a^2x + 3ax^2 - x^3$.

2. Divide $x^4 - 1$ by $x - 1$.

$$\begin{array}{lllll|l}
1 & +0 & +0 & +0 & -1 & 1-1 \\
\cline{6-6}
1 & -1 & & & & 1+1+1+1 \\
\cline{1-2}
 & 1 & +0 & & & \\
 & 1 & -1 & & & \\
\cline{2-3}
 & & 1 & +0 & & \\
 & & 1 & -1 & & \\
\cline{3-4}
 & & & 1 & -1 & \\
 & & & 1 & -1 & \\
\cline{4-5}
\end{array}$$

Ans. $x^3 + x^2 + x + 1$.

3. $(2y^4 - 16y^3 + 2y^2 + 92y + 48) \div (y^2 - 5y - 12) = ?$

4. $(5x^4 - 14x^3y + 31x^2y^2 - 22xy^3 + 12y^4) \div (5x^2 - 4xy + 3y^2) = ?$

5. $(6a^4b^2 + 3a^3b^3 - 4a^2b^4 + b^6) \div (3a^3b - 2ab^3 + b^4) = ?$

This process is called Division by Detached Coefficients. Many of the examples in § 81 will serve as additional exercises in this method.

83. By making use of brackets a neat and concise method is presented for working out certain examples in Multiplication and Division.

1. Multiply $(x+1)^2 + 2(x+1) + 1$ by $(x+1) + 2$.

$$\begin{array}{l}
(x+1)^2 + 2(x+1) + 1 \\
(x+1) + 2 \\
\hline
(x+1)^3 + 2(x+1)^2 + (x+1) \\
\qquad\qquad\;\; 2(x+1)^2 + 4(x+1) + 2 \\
\hline
(x+1)^3 + 4(x+1)^2 + 5(x+1) + 2
\end{array}$$

2. Divide $(x+1)^3 + 4(x+1)^2 + 5(x+1) + 2$ by $(x+1) + 2$.

$$\begin{array}{l}
(x+1)^3 + 4(x+1)^2 + 5(x+1) + 2 \;\big(\,(x+1) + 2 \\
(x+1)^3 + 2(x+1)^2 \qquad\qquad\qquad\;\; (x+1)^2 + 2(x+1) + 1 \\
\hline
\qquad 2(x+1)^2 + 5(x+1) \\
\qquad 2(x+1)^2 + 4(x+1) \\
\hline
\qquad\qquad\qquad (x+1) + 2 \\
\qquad\qquad\qquad (x+1) + 2 \\
\hline
\end{array}$$

3. Multiply $x + a$, $x + b$, and $x + c$, together.

$$\begin{array}{l}
x + a \\
x + b \\
\hline
x^2 + a\,x \\
\quad\; + b\,x + a\,b \\
\hline
x^2 + \;(a+b)\,x \;\; + a\,b \\
x + c \\
\hline
x^3 + \;(a+b)\,x^2 + a\,b\,x \\
\qquad\qquad\quad c\,x^2 + c\,(a+b)\,x + a\,b\,c \\
\hline
x^3 + \;(a+b+c)\,x^2 + (a\,b + a\,c + b\,c)\,x + a\,b\,c
\end{array}$$

4. Divide $x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc$ by $x + c$.

$$
\begin{array}{l|l}
x^3+(a+b+c)x^2+(ab+ac+bc)x+abc & x+c \\
\underline{x^3+\qquad\quad cx^2\qquad\qquad\qquad\qquad} & x^2+(a+b)x+ab \\
\qquad(a+b)x^2+(ab+ac+bc)x & \\
\qquad\underline{(a+b)x^2+\qquad(ac+bc)x\qquad} & \\
\qquad\qquad\qquad\qquad abx+abc & \\
\qquad\qquad\qquad\qquad \underline{abx+abc} &
\end{array}
$$

5. Multiply $x^2 + y^2 + z^2 - yz - zx - xy$ by $x + y + z$.

$$
\begin{array}{l}
x^2-(y+z)x\ +(y^2-yz+z^2) \\
\underline{x\ +\ (y+z)\qquad\qquad\qquad\qquad\qquad} \\
x^3-(y+z)x^2+(y^2-\ \ yz+z^2)x \\
\underline{\quad+(y+z)x^2-(y^2+2yz+z^2)x+(y^3+z^3)} \\
x^3\qquad\qquad\quad -3yzx\qquad\qquad\qquad +\ y^3+z^3
\end{array}
$$

6. Divide $x^3 - 3yzx + y^3 + z^3$ by $x + y + z$.

$$
\begin{array}{l|l}
x^3-3yzx+(y^3+z^3) & x+(y+z) \\
\underline{x^3+(y+z)x^2\qquad\qquad\qquad} & x^2-(y+z)x+(y^2-yz+z^2) \\
\quad-(y+z)x^2-3yzx & \\
\quad\underline{-(y+z)x^2-(y^2+2yz+z^2)x} & \\
\qquad\qquad\qquad(y^2-\ \ yz+z^2)x+(y^3+z^3) & \\
\qquad\qquad\qquad\underline{(y^2-\ \ yz+z^2)x+(y^3+z^3)} &
\end{array}
$$

7. $[x^2 - (b+c)x + bc] \times (x-a) = ?$

8. $[x^3 - (a+p)x^2 + (q+ap)x - aq] \div (x-a) = ?$

9. $[3(x+1)^2 + 4(x+1) - 8] \times [-2(x+1)^2 + 5(x+1) + 1] = ?$

10. $[x^3 + 6xy - (8y^3 - 1)] \div [x - (2y - 1)] = ?$

11. $(1 - x^3 + 8y^3 + 6xy) \div (1 - x + 2y) = ?$

12. $[(x+1)^2 + 3(x+1) + 2] \times [(x+1) + 1] = ?$

13. $[3(x-1) - 5] \times [(x-1) + 1] = ?$

14. $(a^3 - b^3 + c^3 + 3abc) \div (a - b + c) = ?$

CHAPTER VI.

THEOREMS OF DEVELOPMENT.

84. FROM the principles already established we are prepared to demonstrate the following important theorems.

THEOREM I.

85. *The square of the sum of two numbers is equal to the square of the first, plus twice the product of the two, plus the square of the second.*

PROOF. Let a and b represent *any* two numbers. Their sum will be $a + b$; and $(a + b)^2 = (a + b)(a + b)$, which expanded becomes $a^2 + 2ab + b^2$, as will appear from the following process:

$$\begin{array}{l} a + b \\ \underline{a + b} \\ a^2 + ab \\ \quad\;\; \underline{+ ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

According to this theorem find the square of

1. $y + z$.
2. $2x + y$.
   Ans. $4x^2 + 4xy + y^2$.
3. $a + 1$.
4. $a + 3b$.
5. $2x + 3$.
6. $3x + 5y$.
   Ans. $9x^2 + 30xy + 25y^2$.
7. $x^2 + y^2$.
8. $a^3 + b^3$.
9. $x + 2$.
10. $2a^3 + 3b^2$.

THEOREM II.

86. *The square of the difference of two numbers is equal to the square of the first, minus twice the product of the two, plus the square of the second.*

PROOF. Let a and b represent the two numbers. Their difference will be $a - b$; and $(a - b)^2 = (a - b)(a - b)$, which expanded becomes $a^2 - 2ab + b^2$, as will appear from the following process:

$$\begin{array}{l} a - b \\ \underline{a - b} \\ a^2 - ab \\ \underline{\quad\; - ab + b^2} \\ a^2 - 2ab + b^2 \end{array}$$

According to this theorem find the square of

1. $c - d$.
2. $2x - 2y$.
   Ans. $4x^2 - 8xy + 4y^2$.
3. $x - 3$.
4. $1 - x$.
5. $x^2 - y^2$.
6. $a - 3b$.
7. $3x - y$.
   Ans. $9x^2 - 6xy + y^2$.
8. $a^3 - b^3$.
9. $x - abc$.
10. $9x - 2y$.

THEOREM III.

87. *The product of the sum and difference of two numbers is equal to the difference of their squares.*

PROOF. Let a and b represent any two numbers. Their sum will be $a + b$, and their difference $a - b$; and $(a + b)(a - b) = a^2 - b^2$, as will appear from the following process:

$$\begin{array}{l} a + b \\ \underline{a - b} \\ a^2 + ab \\ \underline{\quad\; - ab - b^2} \\ a^2 \qquad\quad - b^2 \end{array}$$

According to this theorem multiply

1. $x + a$ by $x - a$.
2. $x + 1$ by $x - 1$.
3. $1 + 2x$ by $1 - 2x$. Ans. $1 - 4x^2$.
4. $a^2 + b^2$ by $a^2 - b^2$.
5. $a + 3b$ by $a - 3b$.
6. $2ab + 3cd$ by $2ab - 3cd$.
7. $5m^2n + 2xy$ by $5m^2n - 2xy$.

88. This theorem suggests an easy method of squaring numbers.

For, since $a^2 = (a + b)(a - b) + b^2$,

$$99^2 = (99 + 1)(99 - 1) + 1^2 = 100 \times 98 + 1 = 9801$$

In accordance with this principle, find the square of

1. 97.

$$97^2 = (97 + 3)(97 - 3) + 3^2 = 100 \times 94 + 9 = 9409$$

2. 95. 3. 498. 4. 45. 5. 995.

THEOREM IV.

89. *The square of a polynomial is equal to the sum of the squares of all its terms, together with twice the product of each term into each of the terms that follow it.*

Proof. Let $a + b + c +$ etc. be any polynomial. Then

$$(a + b + c + \text{etc.})^2 = (a + b + c + \text{etc.})(a + b + c + \text{etc.}),$$

which expanded becomes

$$a^2 + b^2 + c^2 + \text{etc.} + 2ab + 2ac + \text{etc.} + 2bc + \text{etc.},$$

as will appear from the following process:

$$\begin{array}{rrrl}
a\ + & b\ + & c\ + & \text{etc.} \\
a\ + & b\ + & c\ + & \text{etc.} \\
\hline
a^2 + & a\,b\ + & a\,c\ + & \text{etc.} \\
+ & a\,b\ + & b^2 + b\,c\ + & \text{etc.} \\
 & + & a\,c + b\,c + c^2 + & \text{etc.} \\
\hline
a^2 + & b^2 + & c^2 + & \text{etc.} \\
+ & 2\,a\,b + & 2\,a\,c + & \text{etc.} \\
 & + & 2\,b\,c + & \text{etc.}
\end{array}$$

According to this theorem find the square of

1. $a + b + c$.

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,a\,b + 2\,a\,c + 2\,b\,c$$

2. $a + b - c$.

$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2\,a\,b - 2\,a\,c - 2\,b\,c$$

3. $a + 2\,b - 3\,c$.

$$(a + 2\,b - 3\,c)^2 = a^2 + (2\,b)^2 + (-3\,c)^2 + 2\,a\,(2\,b) + 2\,a\,(-3\,c) + 4\,b\,(-3\,c)$$
$$= a^2 + 4\,b^2 + 9\,c^2 + 4\,a\,b - 6\,a\,c - 12\,b\,c$$

4. $x - 2\,y - 3\,z$.

5. $x^2 - y^2 - z^2$.

6. $1 + 2\,x - 3\,x^2$. Ans. $1 + 4x^2 + 9x^4 + 4\,x - 6x^2 - 12x^3$.

7. $a - b - c$.

8. $a + 2\,b + c$.

9. $x - y + a - b$.

10. $m - n - p - q$.

11. $c + d - x + y$.

12. $3 + x - y + z$.

THEOREM V.

90. *The product of two binomials of the form of $x + a$, $x + b$, is equal to the square of the first term, plus the sum of the second terms into x, plus the product of the second terms.*

PROOF. Let $x + a$, $x + b$, represent the binomials. Then

$$(x + a)\ (x + b) = x^2 + (a + b)\ x + a\,b,$$

as will appear from the following process:

$$\begin{array}{l} x + a \\ x + b \\ \hline x^2 + a\,x \\ \quad + b\,x + a\,b \\ \hline x^2 + (a + b)\,x + a\,b \end{array} \qquad (1)$$

91. This includes all possible cases. For, putting in (1),

$-a$, $-b$, for a and b,

then, $-a$, for a,

and finally, $-b$, for b,

we get the following three additional cases:

$$\left.\begin{array}{llll} & \{x + (-a)\} & \{x + (-b)\} & = x^2 + (-a - b)\,x + a\,b \\ \text{or,} & (x - a) & (x - b) & = x^2 - (a + b)\,x + a\,b \end{array}\right\} (2)$$

$$\left.\begin{array}{llll} & \{x + (-a)\} & (x + b) & = x^2 + (-a + b)\,x - a\,b \\ \text{or,} & (x - a) & (x + b) & = x^2 - (a - b)\,x - a\,b \end{array}\right\} (3)$$

$$\left.\begin{array}{llll} & (x + a) & \{x + (-b)\} & = x^2 + (a - b)\,x - a\,b \\ \text{or,} & (x + a) & (x - b) & = x^2 + (a - b)\,x - a\,b \end{array}\right\} (4)$$

92. The following examples illustrate these four cases.

(1.)

$$\begin{array}{l} x + 9 \\ x + 7 \\ \hline x^2 + 9\,x \\ \quad + 7\,x + 63 \\ \hline x^2 + 16\,x + 63 \end{array}$$

(2.)

$$\begin{array}{l} x - 9 \\ x - 7 \\ \hline x^2 - 9\,x \\ \quad - 7\,x + 63 \\ \hline x^2 - 16\,x + 63 \end{array}$$

(3.)

$$\begin{array}{l} x - 9 \\ x + 7 \\ \hline x^2 - 9\,x \\ \quad + 7\,x - 63 \\ \hline x^2 - 2\,x - 63 \end{array}$$

(4.)

$$\begin{array}{l} x + 9 \\ x - 7 \\ \hline x^2 + 9\,x \\ \quad - 7\,x - 63 \\ \hline x^2 + 2\,x - 63 \end{array}$$

According to this theorem, multiply

5. $x + 8$ by $x + 5$.

6. $x - 8$ by $x - 5$. Ans. $x^2 - 13x + 40$.

7. $x + 8$ by $x - 5$.

8. $x - 8$ by $x + 5$.

9. $a - 3$ by $a + 12$.

10. $x - 4y$ by $x - 10y$.

11. $x - 3a$ by $x + 2a$. Ans. $x^2 - ax - 6a^2$.

12. $a - 1$ by $a + 1$.

13. $x + 6$ by $x - 1$.

14. $x + 6a$ by $x - 5a$.

15. $a - 5b$ by $a + 10b$.

16. $y + 4x$ by $y - 5x$.

93. MISCELLANEOUS EXAMPLES.

1. Find the square of $3ax + 2by$.
Ans. $9a^2x^2 + 12abxy + 4b^2y^2$.

2. Find the square of $5abc - 7abc$.

3. Expand $(5abc - c)^2$.

4. Multiply $2a^2 - 3b$ by $2a^2 + 3b$.

5. Multiply $3x + 7a^2b$ by $3x - 7a^2b$.

6. Expand $(5x^5 - 4y^4)^2$.

7. Expand $(4a^2b - 5ab^3)^2$.

8. Find the square of $xy + yz + zx$.
Ans. $x^2y^2 + y^2z^2 + x^2z^2 + 2xy^2z + 2x^2yz + 2xyz^2$.

9. Find the product of $a^{16} + b^{16}$, $a^8 + b^8$, $a^4 + b^4$, $a^2 + b^2$, $a + b$, and $a - b$. Ans. $a^{32} - b^{32}$.

10. Find the product of $1 + a$, $1 - a$, $1 + a^2$, $1 - a^4$.

11. Find the product of $(a + b + c)(a + b - c)$.

12. Find the product of $(a - b + c)(b + c - a)$.

13. Find the product of $(a^2 + a + 1)(a^2 - a + 1)$.

14. Find the product of $(a + b)^2 (a^2 - 2ab - b^2)$.

15. Expand $(a + b - c + e)^2$.

CHAPTER VII.

FACTORING.

94. An algebraic expression which contains no terms in the fractional form is called an *integral expression.*

Thus, $x^2 - y^2$, $2\,ax - 3\,b$, are integral expressions.

An expression is *rational* when none of its terms contain square or other roots.

95. The **Factors** of such expressions are the rational and integral expressions whose product will produce these expressions.

96. A **Prime Factor** is one that is divisible without a remainder by no rational and integral expression except $\pm$ itself and ± 1.

97. The factors of a purely algebraic monomial are apparent.

Thus, the factors of a^2bxyz are a, a, b, x, y, and z.

98. Polynomials are factored in accordance with the principles of division and the theorems of the preceding chapter.

Case I.

99. When all the Terms have a Common Factor.

1. Find the factors of $ax - ay + az$.

$$(ax - ay + az) = a\,(x - y + z)$$

As a is a factor of each term, it must be a factor of the polynomial; and if we divide the polynomial by a, we obtain the other factor. Hence the following

Rule.

Divide the given polynomial by the common factor; take the quotient thus obtained for one of the factors, and the divisor for the other.

NOTE. The greatest monomial factor is usually sought. The two factors may often be still further resolved.

Find the factors of:

2. $6\,x\,y - 36\,x\,y^2 - 24\,a\,x^2\,y^3$.
Ans. 2, 3, x, y, and $1 - 6\,y - 4\,a\,x\,y^2$.

3. $3\,a^2 - 6\,a\,b$. Ans. 3, a, and $a - 2\,b$.

4. $a^3 - a\,x$.

5. $5\,a^2\,b\,x^3 - 15\,a\,b\,x^2 - 20\,b^3\,x^2$.

6. $x^3 - x^2\,y + x\,y^2$.

7. $a^3\,b - 2\,a^2\,b^2 - 2\,a\,b^3$.

8. $38\,a^3\,x^5 + 57\,a^4 x^2$. Ans. 19, a^3, x^2, and $2\,x^3 + 3\,a$.

9. $3\,x^4\,y^3\,z^2 - 6\,x^2\,y^4\,z^3 + 12\,x^3\,y^2\,z^4$.

10. $x^n\,y^n + x^{n+p}\,y^{n+q}$.

11. $7\,a\,y + 5\,x y^2 - 10\,a\,x^2$.

12. $a\,x^{m+2}\,y^{n+3} + b\,x^{m+1}\,y^{n+2} + c\,x^m\,y^{n+1}$.
Ans. x^m, y^{n+1}, and $a\,x^2 y^2 + b x\,y + c$.

13. $x^{2a+b+c} + x^{a+2b+c} + x^{a+b+2c}$.

14. $12\,a^2\,x\,y^3 - 18\,a^3\,x^2\,y + 24\,a\,x^3\,y^2$.

CASE II.

100. When one Term of a Trinomial is equal to twice the Product of the Square Roots of the other two.

1. Find the factors of $x^2 + 2\,x\,y + y^2$.

$$x^2 + 2\,x y + y^2 = (x + y)\,(x + y)$$

We resolve this into its factors at once by the converse of the principle in Theorem I. § 85.

2. Find the factors of $x^2 - 2xy + y^2$.

$$x^2 - 2xy + y^2 = (x - y)(x - y)$$

We resolve this into its factors at once by the converse of the principle in Theorem II. § 86. Hence the following

Rule.

Omitting the term that is equal to twice the product of the square roots of the other two, take for each factor the square root of each of the other two connected by the sign of the term omitted.

Find the factors of:

3. $c^2 + 2cd + d^2$.
4. $x^2 + 14x + 49$.
5. $x^2 + 6x + 9$.
6. $x^2 - 22x + 121$.
7. $1 - 6x + 9x^2$.
8. $x^2 - 6xy + 9y^2$.
9. $x^4 - 2a^2x^3 + a^4x^2$.
10. $4 + 9x^2 - 12x$.
11. $6a^3x + a^2x^2 + 9a^4$.
12. $x^4 - 4x^2 + 4$.
13. $25y^2 + 1 - 10y$.
14. $30x^2y + 3x^4 + 75y^2$.
15. $2x^{10}y^4 - 60x^5y^2 + 450$.
16. $(a+b)^2 + 2(ac+bc) + c^2$.
17. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$. Ans. $(x+y+z)^2$.

NOTE. This and the following examples can be written, like the 16th, as binomials.

18. $a^2 - 2ab + b^2 + 2bc + c^2 - 2ac$.
19. $x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4$.
20. $x^4 - 2x^3 + 3x^2 - 2x + 1$.

CASE III.

101. When a Binomial is the Difference between Two Squares.

1. Find the factors of $x^2 - y^2$.

$$x^2 - y^2 = (x + y)(x - y)$$

We resolve this into its factors at once by the converse of the principle in Theorem III. § 87. Hence the following

Rule.

Take for one of the factors the sum, and for the other the difference, of the square roots of the terms of the binomial.

Find the factors of:

2. $x^2 - 9$.
3. $x^2 - 25$.
4. $4 - x^2$.

   Ans. $2 + x$ and $2 - x$.
5. $81 - x^2$.
6. $x^2 - 9a^2$.
7. $x^2 - 16a^2$.
8. $9x^2 - 25y^2$.
9. $49x^2 - 4y^2$.
10. $1 - 9a^2$.
11. $121a^2 - 1$.

    Ans. $11a + 1$, and $11a - 1$.
12. $49x^4 - 81a^2$.
13. $x^6y^8 - 4a^2$.
14. $a^2b^4c^6 - x^{14}$.
15. $36x^{36} - 49a^{18}$.
16. $1 - 81a^2c^2d^2$.
17. $16x^{16} - 4y^4$.
18. $a^4b^6c^2 - x^2y^4z^6$.
19. $121a^4 - 49b^6$.
20. $3 - 12x^2$.

    Ans. $3, 1+2x$, and $1-2x$.
21. $8 - 50a^2b^2$.
22. $2a^4b^{12} - 8c^8$.
23. $2x^{16} - 512$.

102. When one or both of the squares is a polynomial, the same method is employed.

1. Find the factors of $9a^2 - (b - c)^2$.

   The square root of $9a^2 = 3a$.

   The square root of $(b - c)^2 = b - c$.

   Their sum is $3a + (b - c) = 3a + b - c$.

   Their difference is $3a - (b - c) = 3a - b + c$.

   Therefore $9a^2 - (b - c)^2 = (3a + b - c)(3a - b + c)$.

Find the factors of:

2. $a^2-(b-c)^2$.

3. $a^2-(b+c)^2$.

4. $(b-c)^2-a^2$.

Ans. $b-c+a$ and $b-c-a$.

5. $(b+c)^2-a^2$.

6. $(a-b)^2-(c-d)^2$.

7. $(a-b)^2-(c+d)^2$.

8. $(a+b)^2-(c-d)^2$.

9. $(x+y)^2-4z^2$.

10. $4a^2-(b-c)^2$.

11. $9x^2-(3a-2b)^2$.

12. $1-(a+b)^2$.

13. $(x+3y)^2-1$.

14. $(a+2b)^2-(3x+5y)^2$.

15. $1-(5a-2b)^2$.

16. $(a-3x)^2-16y^2$.

17. $(2a-3b)^2-1$.

18. $(2a-3b)^2-(c+d-2y)^2$.

19. $(a+b-c)^2-(x+y-z)^2$.

Resolve into factors and simplify:

20. $(3x+7y)^2-(2x-3y)^2$.

$$\begin{aligned}&(3x+7y)^2-(2x-3y)^2\\&=(3x+7y+2x-3y)(3x+7y-2x+3y)\\&=(5x+4y)(x+10y)\end{aligned}$$

21. $(x-y)^2-(x+y)^2$.

22. $(x+y)^2-(x-y)^2$.

23. $(5x+2y)^2-(3x-y)^2$.

24. $9x^2-(3x-5y)^2$.

25. $16a^2-(3a+1)^2$.

26. $(3a+1)^2-(2a-1)^2$.

27. $(2a+b-c)^2-(a+b+c)^2$.

28. $(2x+a-3)^2-(3-2x)^2$.

29. $(x+y-4)^2-(x-4)^2$.

30. $(a+b+c)^2-(a-b-c)^2$.

103. Polynomials may often be arranged in two groups with the minus sign between them, and so be factored as above.

Find the factors of:

1. $a^2 - 2ab + b^2 - c^2$.

$$\begin{aligned} &a^2 - 2ab + b^2 - c^2 \\ &= (a-b)^2 - c^2 \\ &= (a-b+c)(a-b-c) \end{aligned}$$

2. $2xy - x^2 - y^2 + z^2$.

$$\begin{aligned} &2xy - x^2 - y^2 + z^2 \\ &= z^2 - (x^2 - 2xy + y^2) \\ &= z^2 - (x-y)^2 \\ &= (z+x-y)(z-x+y) \end{aligned}$$

3. $x^2 + a^2 + 2ax - y^2$.

Ans. $x + a + y$, and $x + a - y$.

4. $1 - x^2 - 2xy - y^2$.

5. $x^2 - 6ax + 9a^2 - 16b^2$.

6. $a^2 - 9b^2 - 2ax + x^2$.

7. $x^2 - 4xy - 9x^2y^2 + 4y^2$.

Ans. $x - 2y + 3xy$, and $x - 2y - 3xy$.

8. $2ab - 1 + a^2 + b^2$.

9. $a^2 - c^2 + b^2 - d^2 - 2ab - 2cd$.

10. $a^2 + 2an + n^2 - b^2 - 2bm - m^2$.

Ans. $a + n + b + m$, and $a + n - b - m$.

11. $x^2 + 2x + 1 - a^2 + 2ax - x^2$.

12. $25b^2 - 1 - 9b^2x^2 - 10ab + a^2 + 6bx$.

13. $1 - 4x + 4x^2 - 1 + 6x - 9x^2$.

14. $4xy - 4x^2 + 1 - y^2$.

15. $4x^2y^2 - x^4 - y^4 - z^4 - 2x^2y^2 + 2x^2z^2 + 2y^2z^2$.

104. Trinomials of the form $x^{4n} + x^{2n} y^{2n} + y^{4n}$ can be written as the difference of two squares, and factored by the above method.

1. Find the factors of $x^4 + x^2 y^2 + y^4$.

$$\begin{aligned} x^4 + x^2 y^2 + y^4 &= x^4 + 2 x^2 y^2 + y^4 - x^2 y^2 \\ &= (x^2 + y^2)^2 - x^2 y^2 \\ &= (x^2 + y^2 + x y)(x^2 + y^2 - x y) \end{aligned}$$

Find the factors of:

2. $a^8 + a^4 b^4 + b^8$. Ans. $a^4 + b^4 + a^2 b^2$, and $a^4 + b^4 - a^2 b^2$.

3. $x^4 - 18 x^2 y^2 + y^4$.

4. $x^4 + x^2 + 1$.

5. $x^4 - 5 x^2 + 4$.

6. $x^4 - 3 x^2 y^2 + y^4$.

7. $x^4 + (2 - m^2) x^2 y^2 + y^4$.

8. $(a + b)^8 + (a + b)^4 + 1$.

9. $x^4 + 7 x^2 + 64$. Ans. $x^2 + 3 x + 8$, and $x^2 - 3 x + 8$.

10. $9 x^4 + 3 x^2 y^2 + 4 y^4$.

11. $x^4 - 171 x^2 + 1$.

12. $16 x^4 + 23 x^2 y^2 + 9 y^4$.

13. $49 x^4 - 74 x^2 y^2 + 25 y^4$.

14. $a^4 b^4 + a^2 b^2 c^2 d^2 + c^4 d^4$.

15. $4 a^8 - 21 a^4 b^4 + 9 b^8$.

Case IV.

105. When the Polynomials can be arranged in Groups of two or more Terms, having a Factor common to all the Groups.

1. Find the factors of $x^2 - a x + b x - a b$.

$$\begin{aligned} x^2 - a x + b x - a b &= (x^2 - a x) + (b x - a b) \\ &= x (x - a) + b (x - a) \\ &= (x - a)(x + b) \end{aligned}$$

Or,

$$\begin{aligned} x^2 - a x + b x - a b &= (x^2 + b x) - (a x + a b) \\ &= x (x + b) - a (x + b) \\ &= (x + b)(x - a) \end{aligned}$$

Hence the following

Rule.

Group the terms of the expression so that each group shall have a monomial factor, then factor each group according to Case I., and finally divide by the factor common to all the groups.

This common factor, and the quotient obtained by the division, will be the factors required.

Find the factors of:

2. $ac - ad + bc - bd$. Ans. $a + b$, and $c - d$.
3. $ax^3 + x^2 + ax + 1$.
4. $mx - my - nx + ny$.
5. $5a + ab + 5b + b^2$.
6. $3ax - bx - 3ay + by$.
7. $2x^2 + 4ax + 6bx + 12ab$.
   Ans. $2x + 6b$, and $x + 2a$.
8. $8a^2 + 12ax + 10ab + 15bx$.
9. $ax^2 - 3bxy - axy + 3by^2$.
10. $2x^4 - x^3 + 4x - 2$.
11. $y^3 - y^2 + y - 1$.
12. $2ax^2 + 3axy - 2bxy - 3by^2$.
13. $x^4 - 2x + x^3 - 2$.
14. $amx^2 + bmxy - anxy - bny^2$.
15. $x^2 - 3x - xy + 3y$.
16. $ax - bx + by + cy - cx - ay$.
    Ans. $x - y$, and $a - b - c$.
17. $a^2x + abx + ac + aby + b^2y + bc$.
18. $a^3 - a^2b + ab^2 - b^3$.

19. $2a + (a^2 - 4)x - 2ax^2$.

20. $xy(a^2 + b^2) - ab(x^2 + y^2)$.

Ans. $ax - by$, and $ay - bx$.

21. $xy(1 + z^2) + z(x^2 + y^2)$.

22. $2x^5 + 2x^4 + 3x^3 + 3x^2 + 4x + 4$.

23. $a^5 - a^4 b + a^3 b^2 - a^2 b^3 + a b^4 - b^5$.

24. $a^3 + 3a^2 b + 3ab^2 + b^3$.

25. $a^3 - 3a^2 + 3a - 1$.

26. $a^2 b - a^2 c - ab^2 + ac^2 + b^2 c - bc^2$.

CASE V.

106. A Trinomial in the form, $x^2 + (a + b)x + ab$, can be separated into two Binomial Factors.

From the converse of Theorem V. § 90,

$$x^2 + (a + b)x + ab = (x + a)(x + b) \ldots \quad (1)$$
$$x^2 - (a + b)x + ab = (x - a)(x - b) \ldots \quad (2)$$
$$x^2 - (a - b)x - ab = (x - a)(x + b) \ldots \quad (3)$$
$$x^2 + (a - b)x - ab = (x + a)(x - b) \ldots \quad (4)$$
$$x^2 + 16x + 63 = (x + 9)(x + 7) \ldots \quad (5)$$
$$x^2 - 16x + 63 = (x - 9)(x - 7) \ldots \quad (6)$$
$$x^2 - 2x - 63 = (x - 9)(x + 7) \ldots \quad (7)$$
$$x^2 + 2x - 63 = (x + 9)(x - 7) \ldots \quad (8)$$

By inspecting the above results, we find that, when a trinomial is in this form,

1. The first term of *both* factors, in the trinomial, is the square root of the first term of the trinomial.

2. The second terms of the factors are such numbers that their product always equals the last term, and their sum the coefficient of the second term.

Hence, for the factoring of a trinomial of this form, we have the following

Rule.

Find two numbers such that their product shall equal the last term of the trinomial, and their sum the coefficient of the second term; join each number, respectively, with its proper sign, to the square root of the first term for the factors required.

1. Find the factors of $x^2 + 6x + 8$.

The second terms of the factors must be such that their product is $+8$, and their sum $+6$. The only pairs of integral numbers that multiplied together make $+8$ are ± 8 and ± 1, ± 4 and ± 2. From these we are to select that pair whose sum is $+6$. These are 4 and 2.

The first term of both factors is the square root of x^2.

$$\therefore\ x^2 + 6x + 8 = (x + 4)(x + 2)$$

2. Find the factors of $x^2 + 4x - 12$.

The only pairs of integral numbers that, multiplied together, make -12 are ± 12 and ∓ 1, ± 6 and ∓ 2, ± 4 and ∓ 3. The pair whose sum is $+4$ is $+6$ and -2. The square root of x^2 is x.

$$\therefore\ x^2 + 4x - 12 = (x + 6)(x - 2)$$

Find the factors of:

3. $x^2 + 3x + 2$.
4. $x^2 + 9x + 20$.
5. $x^2 - 9x + 14$.
6. $x^2 - 3x - 10$.
7. $x^2 + 2x - 35$.
8. $x^2 + 17x + 72$.
9. $x^2 + 23x + 22$.
10. $x^2 - 5xy + 4y^2$.
11. $x^2 + 11xy + 30y^2$.
12. $x^2 - 9xy + 20y^2$.

13. $x^2 - x - 132$.

14. $x^2 - x - 6$.

15. $x^2 - x - 2$.

16. $x^2 + x - 6$.

17. $x^2 + x - 2$.

18. $x^2 y^2 - 5 x y - 24$.

19. $x^4 - a^2 x^2 - 132 a^4$.

20. $x^2 - 32 x y - 105 y^2$.

21. $x^2 y^2 - 21 x y + 110$.

22. $x^2 + 21 x - 100$.

23. $x^2 + 29 x y - 30 y^2$.

24. $1 - 7 x - 30 x^2$.

25. $x^6 - x^3 - 110$.

26. $12 - 7 x + x^2$.

27. $8 x^2 + x^4 + 7$.

28. $a - 20 + a^2$.

29. $6 + x - x^2$. Write it, $-1 (x^2 - x - 6)$.

30. $98 - 7 x - x^2$.

31. $65 + 8 x y - x^2 y^2$.

32. $10 x^2 + 3 x - 1$.

33. $1 + 4 x - 96 x^2$.

34. $a^2 - 18 a x y - 243 x^2 y^2$.

35. $x^4 + 13 a^2 x^2 - 300 a^4$.

36. $x^4 - a^2 x^2 - 462 a^4$.

37. $120 - 7 a x - a^2 x^2$.

38. $x^2 y^2 z^2 + 16 x y z - 260$.

39. $x^3 - 3 x^2 - 18 x$.

40. $x^3 y - x^2 y^2 - 2 x y^3$.

41. $3 x^2 + 51 x y + 90 y^2$.

42. $x^6 + x^3 - 870$.

43. $11 x + 152 - x^2$.

44. $x^2 + (a + 2 b) x + 2 a b$.

45. $x^2 + (a + 2 b) x + (a b + b^2)$.

46. $x^{2n} + (a + b) x^n y^n + a b y^{2n}$.

47. $x^2 + 2 (a^2 + b^2) x + (a^2 - b^2)^2$.

48. $x^2 - (3 b c + c a + a b) x + 3 b c (b + c) a$.

49. $6 a b x^2 y^2 z^2 - 42 a b x y z - 108 a b$.

50. $13 m^6 n^6 - 221 m^3 n^3 x^3 + 234 x^6$.

51. $3 m^2 x^8 + 66 m x^4 y^3 - 225 y^6$.

52. $7 b^2 x^2 + 14 b d x y - 105 d^2 y^2$.

107. If the coefficient of the highest power is not unity, and is not a common factor, we can still divide the expression by this coefficient, and then factor the quotient as in the last article.

1. Find the factors of $3x^2 + 14x + 8$.

$$3x^2 + 14x + 8 = 3\left(x^2 + \tfrac{14}{3}x + \tfrac{8}{3}\right) = 3\left(x^2 + \tfrac{14}{3}x + \tfrac{24}{9}\right)$$

The first term of each of the two factors of $x^2 + \frac{14}{3}x + \frac{24}{9}$ must be x. Of the second terms $+\frac{24}{9}$ is the product, and $+\frac{14}{3}$ is the sum.

Hence the second terms must be $\frac{2}{3}$, and $\frac{12}{3}$, or 4.

$$\therefore\ 3\left(x^2 + \tfrac{14}{3}x + \tfrac{24}{9}\right) = 3\left(x + \tfrac{2}{3}\right)(x + 4) = (3x + 2)(x + 4)$$

The denominators of the second terms of the two factors are always the same as the denominator of the *sum*; in this instance, 3. The numerators are obtained from the numerators of the sum and product, precisely as in the preceding article; in this instance 12 and 2, from 14 and 24, regarded respectively as the sum and product.

Further, it is evident that the denominator of the fraction which is the product of the second terms must be the square of the denominator of the fraction which is the sum of these terms. Thus above we must use $\frac{24}{9}$ instead of $\frac{8}{3}$.

The process, without explanation, will appear as follows:

$$\begin{aligned} 3x^2 + 14x + 8 &= 3\left(x^2 + \frac{14}{3}x + \frac{24}{(3)^2}\right) \\ &= 3\left(x + \tfrac{2}{3}\right)(x + 4) \\ &= (3x + 2)(x + 4) \end{aligned}$$

2. Find the factors of $3x^2 - 10x + 8$.

$$\begin{aligned} 3x^2 - 10x + 8 &= 3\left(x^2 - \frac{10}{3}x + \frac{24}{(3)^2}\right) \\ &= 3\left(x + \tfrac{2}{3}\right)(x - 4) \\ &= (3x + 2)(x - 4) \end{aligned}$$

3. Find the factors of $7x^2 - 19x - 6$.

$$7x^2 - 19x - 6 = 7\left(x^2 - \frac{19}{7}x - \frac{42}{(7)^2}\right)$$
$$= 7\left(x + \tfrac{2}{7}\right)(x - 3)$$
$$= (7x + 2)(x - 3)$$

4. Find the factors of $10x^2 - 13x - 3$.

$$10x^2 - 13x - 3 = 10\left(x^2 - \frac{13}{10}x - \frac{30}{(10)^2}\right)$$
$$= 5\left(x + \tfrac{2}{10}\right)2\left(x - \tfrac{15}{10}\right)$$
$$= (5x + 1)(2x - 3)$$

5. Find the factors of $2x^2 - 5xy - 3y^2$.

$$2x^2 - 5xy - 3y^2 = 2\left(x^2 - \frac{5y}{2}x - \frac{6y^2}{(2)^2}\right)$$
$$= 2\left(x + \frac{y}{2}\right)(x - 3y)$$
$$= (2x + y)(x - 3y)$$

Find the factors of:

6. $3x^2 + 5x + 2$.

Ans. $x + 1$, and $3x + 2$.

7. $2x^2 + 11x + 5$.

8. $4x^2 + 11x - 3$.

9. $3x^2 + 14x - 5$.

10. $6x^2 - 31x + 35$.

11. $3 + 11x - 4x^2$.

Ans. $1 + 4x$, and $3 - x$.

12. $2 - 3x - 2x^2$.

13. $3x^2 + 19x - 14$.

14. $2x^2 + 15x - 8$.

15. $12x^2 - 23xy + 10y^2$.

16. $24x^2 - 29xy - 4y^2$.

Ans. $3x - 4y$, and $8x + y$.

17. $6x^2 + 31x + 35$.

18. $20 - 9x - 20x^2$.

19. $21x^2 + 26xy - 15y^2$.

Ans. $(7x - 3y)(3x + 5y)$.

20. $ax^2 + (a + b)x + b$.

21. $ax^2 + (a - b)x - b$.

22. $abx^2 - (a^2 - b^2)x - ab$.

Ans. $(ax + b)(bx - a)$.

Case VI.

108. When the Expression is a Binomial of the Form, $a^n \pm b^n$, n being a Positive Integer.

(1) $a+b$ is a factor of a^n+b^n when n is odd, but not when n is even.
(2) $a+b$ is a factor of a^n-b^n when n is even, but not when n is odd.
(3) $a-b$ is a factor of a^n-b^n always.
(4) $a-b$ is a factor of a^n+b^n never. (See Preface.)

I. To prove (1):

It is evident that at each successive step of the division of a^n+b^n by $a+b$, the exponent of a in the successive remainders will diminish by one, and hence eventually become zero.

At this stage of the process, let Q represent the quotient and R the remainder, if any. R will not involve a, as $a^0=1$. The product of the divisor by the quotient plus the remainder equals the dividend.

$$\therefore\ Q(a+b)+R=a^n+b^n$$

Now the equation must be true whatever value we assign to a, and R will remain unchanged, since it does not involve a.

Let $\qquad a=-b$

then $\qquad Q(-b+b)+R=(-b)^n+b^n$

but $\qquad Q(-b+b)=0$

$$\therefore\ R=(-b)^n+b^n$$

But $(-b)^n+b^n=0$, when n is odd, and $2b^n$, when n is even; showing that $a+b$ is a factor of a^n+b^n when n is odd, but not when n is even.

II. To prove (3):

As before, when the exponent of a becomes zero in the division, let Q represent the quotient and R the remainder.

Then $\qquad Q(a-b)+R=a^n-b^n$

Substituting a for b in the equation, and recollecting that, since R does not involve a, it will remain unchanged, we have

$$Q(a-a)+R=a^n-a^n$$

From which it is at once seen that $R = 0$, whether n is odd or even, and that hence $a - b$ must be a factor of $a^n - b^n$, whether n is odd or even.

The proofs of statements (2) and (4) are reserved as exercises for the student.

The *law of the formation* of the quotients, or second factors, is simple, and may be determined by actual division, thus:

$$
\begin{array}{l}
a + b)\ a^n + b^n \quad (a^{n-1} - a^{n-2}b \ldots\ldots + b^{n-1} \\
\qquad\quad \underline{a^n + a^{n-1}b} \\
\qquad\qquad - a^{n-1}b + b^n \\
\qquad\qquad \underline{- a^{n-1}b - a^{n-2}b^2} \\
\qquad\qquad\qquad\qquad a^{n-2}b^2 + b^n \\
\qquad\qquad\qquad\qquad \underline{\ldots\ldots\ldots} \\
\qquad\qquad\qquad\qquad\qquad + a\,b^{n-1} + b^n \\
\qquad\qquad\qquad\qquad\qquad \underline{+ a\,b^{n-1} + b^n}
\end{array}
$$

The following are the *general expressions* for the factors of $a^n + b^n$, and $a^n - b^n$, when n is *odd:*

(1) $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 \ldots\ldots - a\,b^{n-2} + b^{n-1})$.

(2) $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 \ldots\ldots + a\,b^{n-2} + b^{n-1})$.

1. Factor $a^5 + b^5$.

Substituting 5 for n in (1), we have

$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - a\,b^3 + b^4).$$

2. Factor $a^5 - b^5$.

Substituting 5 for n in (2), we have

$$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + a\,b^3 + b^4).$$

3. Factor $8 + c^3$.

$$8 + c^3 = 2^3 + c^3 = (2 + c)(2^2 - 2c + c^2).$$

Find the factors of:

4. $a^3 + b^3$.
6. $x^7 - y^7$.
8. $343 - z^3$.
5. $a^3 - b^3$.
7. $c^5 + 32$.
9. $a^5 + 243$.

109. The factors of such examples can be written out by inspection. Attention to the following laws will enable one readily to do this:

1. The terms of the quotient, or second factor, are *all positive* when the divisor is $a - b$, and *alternately positive* and *negative* when the divisor is $a + b$.

2. The number of terms always corresponds to the degree of the binomial.

3. a appears in the first term, b in the last term, and a and b in all the intermediate terms.

4. The exponent of a in the first term is one less than the degree of the binomial, and decreases regularly by unity in each successive term; the exponent of b in the second term is 1, and increases regularly by one in each successive term, till in the last term it becomes the same as the exponent of a in the first term.

5. The sum of the exponents of a and b in *any* intermediate term is always the same, and is equal to the exponent of a in the first term. a and b stand for any letters or expressions.

Factor by inspection the following expressions:

1. $c^3 + d^3$.
4. $x^3 + 1$.
7. $3x^3 + 24y^3$.
2. $c^3 - d^3$.
5. $x^3 - 1$.
8. $27x^3 + 1$.
3. $125 + a^3$.
6. $8x^3 - y^3$.
9. $1 - 8y^3$.
10. $27x^3 - 8y^3$.

$$27x^3 - 8y^3 = (3x)^3 - (2y)^3$$
$$= (3x - 2y)(9x^2 + 6xy + 4y^2)$$

11. $16a^3b^3 + 250x^3$.
12. $a^3 + 343b^3$.
13. $216 - a^3$. Ans. $6 - a$, and $36 + 6a + a^2$.
14. $a^3b^3 + 512$.
18. $343 - 8a^3$.
15. $216y^3 - z^3$.
19. $343x^3 + 1000z^3$.
16. $1 - 343x^3$.
20. $x^6 - 27y^3$.
17. $40x^3 - 135y^3$.
21. $a^6 - 729b^6$.

22. $8\,a^3 x^3 - 27\,b^3 y^3$.

23. $27\,m^3 - 64\,n^3$.

24. $125\,x^6 + 64\,y^6$.

25. $(x + y)^3 + (x - y)^3$.

26. $(4\,x^2 - 1)^3 - (4\,x^2 + 1)^3$.

27. $10\,a^3 - 640\,y^6$.

28. $x^3 y^3 - 216\,z^3$.

29. $4\,a^3 b^3 c^3 - 4$.

30. $27\,a^3 - 64\,y^3$.

31. $(x - 2\,y)^3 - (y - 2\,x)^3$.

32. $(x + 2\,y)^3 + (y + 2\,x)^3$.

33. $729\,a^3 - 64\,b^3$.

34. $270 - 10000\,x^3$.

35. $a^3 b^3 - \frac{1}{8}\,c^3$.

36. $27\,x^3 y^3 - \frac{1}{8}\,z^3$.

37. $(a + b)^3 - c^3$.

110. When n is *even* and greater than 2, there will be three or more factors in each case, and they can be more expeditiously determined by Case III., with other principles already explained.

1. Find the factors of $x^4 - y^4$.

$$\begin{aligned} x^4 - y^4 &= (x^2 + y^2)\,(x^2 - y^2) \\ &= (x^2 + y^2)\,(x + y)\,(x - y) \end{aligned}$$

2. Find the factors of $x^6 - y^6$.

$$\begin{aligned} x^6 - y^6 &= (x^3 + y^3)\,(x^3 - y^3) \\ &= (x + y)\,(x^2 - x\,y + y^2)\,(x - y)\,(x^2 + x\,y + y^2) \end{aligned}$$

3. Find the factors of $x^8 - 1$.

$$\begin{aligned} x^8 - 1 &= (x^4 + 1)\,(x^4 - 1) \\ &= (x^4 + 1)\,(x^2 + 1)\,(x^2 - 1) \\ &= (x^4 + 1)\,(x^2 + 1)\,(x + 1)\,(x - 1) \end{aligned}$$

Find the factors of:

4. $x^8 - y^8$.

5. $a^6 - b^6$.

6. $x^4 - 1$.

7. $x^6 - 1$.

8. $1 - x^8$.

9. $1 - x^6$.

10. $1 - x^4$.

11. $x^{10} - y^{10}$.

12. $x^6 - 64$.

13. $x^{16} - y^{16}$.

14. $a^6 b^6 - c^6 d^6$.

15. $a^{12} - a^4$.

16. $a^{11} - a^5$.

17. $a^{12} - b^{12}$.

18. $64\,x^7 - x$.

19. $z^{12} - a^6 b^6$.

20. $a^9 - 729\,a^3$.

21. $x^8 - \frac{1}{256}$.

111. Though $a^n + b^n$ is not divisible by $a + b$ when n is *even*, it is possible to find a binomial factor in every case except when n is a power of 2, such as 2, 4, 8, 16, etc.

1. Find the factors of $a^6 + b^6$.

$$\begin{aligned} a^6 + b^6 &= (a^2)^3 + (b^2)^3 \\ &= (a^2 + b^2)\ \{(a^2)^2 - a^2b^2 + (b^2)^2\} \\ &= (a^2 + b^2)\ (a^4 - a^2b^2 + b^4) \end{aligned}$$

2. Find the factors of $a^{12} + b^{12}$.

$$\begin{aligned} a^{12} + b^{12} &= (a^4)^3 + (b^4)^3 \\ &= (a^4 + b^4)\ \{(a^4)^2 - a^4b^4 + (b^4)^2\} \\ &= (a^4 + b^4)\ (a^8 - a^4b^4 + b^8) \end{aligned}$$

Find the factors of:

3. $a^{10} + b^{10}$.
4. $a^6 + 1$.
5. $x^{12} + 1$.
6. $x^{10} + 1024\,y^{10}$.
7. $64\,x^6 + 1$.
8. $a^{14} + b^{14}$.

112. The examples in all the cases, thus far, save those in Case I, can be factored by Case IV.

1. Factor $x^2 - 2\,x\,y + y^2$ (Case II.).

$$\begin{aligned} x^2 - 2\,x\,y + y^2 &= x^2 - x\,y - x\,y + y^2 \\ &= x\,(x - y) - y\,(x - y) \\ &= (x - y)\ (x - y) \\ &= (x - y)^2 \end{aligned}$$

2. Factor $x^2 - y^2$ (Case III.).

$$\begin{aligned} x^2 - y^2 &= x^2 + x\,y - x\,y - y^2 \\ &= x\,(x + y) - y\,(x + y) \\ &= (x + y)\ (x - y) \end{aligned}$$

3. Factor $x^2 + 6\,x + 8$ (Case V.).

$$\begin{aligned} x^2 + 6\,x + 8 &= x^2 + 2\,x + 4\,x + 8 \\ &= x\,(x + 2) + 4\,(x + 2) \\ &= (x + 2)\ (x + 4) \end{aligned}$$

4. Factor $a^3 + b^3$ (Case VI.).

$$\begin{aligned} a^3 + b^3 &= a^3 + a^2 b - a^2 b - a b^2 + a b^2 + b^3 \\ &= a^2 (a + b) - a b (a + b) + b^2 (a + b) \\ &= (a + b)(a^2 - a b + b^2) \end{aligned}$$

By this method find the factors of the following examples:

5. $x^2 + 2 x y + y^2$.
6. $x^2 - 6 x + 8$.
7. $x^3 - y^3$.
8. $x^2 - 4$.
9. $x^2 - 1$.
10. $4 x^2 + 4 x y + y^2$.
11. $8 - x^3$.
12. $x^2 - x - 6$.
13. $x^3 + x - 10$.
14. $x^4 - 1$.

113. To be expert in factoring, it is necessary to become familiar with the following algebraic expressions:

$a^2 + b^2$ is prime.

$a^2 - b^2 = (a + b)(a - b)$.

$a^3 + b^3 = (a + b)(a^2 - a b + b^2)$.

$a^3 - b^3 = (a - b)(a^2 + a b + b^2)$.

$a^4 + b^4$ is prime.

$a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$.

$a^5 + b^5 = (a + b)(a^4 - a^3 b + a^2 b^2 - a b^3 + b^4)$.

$a^5 - b^5 = (a - b)(a^4 + a^3 b + a^2 b^2 + a b^3 + b^4)$.

$a^5 + b^6 = (a^2 + b^2)(a^4 - a^2 b^2 + b^4)$.

$a^6 - b^6 = (a + b)(a^2 - a b + b^2)(a - b)(a^2 + a b + b^2)$.

$a^7 + b^7 = (a + b)(a^6 - a^5 b + a^4 b^2 - a^3 b^3 + a^2 b^4 - a b^5 + b^6)$.

$a^7 - b^7 = (a - b)(a^6 + a^5 b + a^4 b^2 + a^3 b^3 + a^2 b^4 + a b^5 + b^5)$.

$a^8 + b^8$ is prime.

$a^8 - b^8 = (a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$.

$a^2 + 2ab + b^2 \quad = (a+b)^2.$

$a^2 - 2ab + b^2 \quad = (a-b)^2.$

$a^4 + 2a^2b^2 + b^4 = (a^2+b^2)^2.$

$a^4 - 2a^2b^2 + b^4 = (a^2-b^2)^2 = (a+b)^2(a-b)^2.$

$a^6 + 2a^3b^3 + b^6 = (a^3+b^3)^2 = (a+b)^2(a^2-ab+b^2)^2.$

$a^6 - 2a^3b^3 + b^6 = (a^3-b^3)^2 = (a-b)^2(a^2+ab+b^2)^2.$

1. What does $a^2 + ab + b^2$ suggest?
2. What does $a^2 + 2ab + b^2$ suggest?
3. What does $a^2 - ab + b^2$ suggest?
4. What does $a^2 - 2ab + b^2$ suggest?
5. What does $(a^4+b^4)(a^2+b^2)(a+b)(a-b)$ suggest?

114. Division by Factors.

Divide:

1. $a^2 - x^2$ by $a - x$.
2. $a^3 - x^3$ by $a - x$.
3. $a^5 - x^5$ by $a - x$.
4. $x^3 + y^3$ by $x + y$.
5. $x^5 + y^5$ by $x + y$.
6. $x^7 + y^7$ by $x + y$.
7. $x - y$ by $y - x$.
8. $x^2 - y^2$ by $y - x$.
9. $x^3 - y^3$ by $x^2 + xy + y^2$.
10. $x^3 - y^3$ by $y - x$.
11. $x^5 - y^5$ by $y - x$.
12. $4x^2 - 1$ by $2x - 1$.
13. $9x^2 - 1$ by $3x + 1$.
14. $25x^2y^2 - 9$ by $5xy - 3$.
15. $8x^3 - 1$ by $2x - 1$.
16. $x^2 - 9$ by $x + 3$.
17. $1 + 27x^3$ by $1 + 3x$.
18. $x^3 + 27$ by $x + 3$.
19. $x^3 - 8y^3$ by $x - 2y$.
20. $x^2 - 4$ by $x + 2$
21. $x^2 + 6x + 5$ by $x + 5$.
22. $x^2 - 8x + 12$ by $x - 6$.
23. $x^2 - 6ax + 9a^2$ by $x - 3a$.
24. $x^2 - 2x + 1$ by $x - 1$.

25. $4x^2 + 12xy + 9y^2$ by $2x + 3y$.

26. $x^3 - 3x^2a + 3xa^2 - a^3$ by $x^2 - 2ax + a^2$.

27. $x^3 + 3x^2 + 3x + 1$ by $x + 1$.

28. $(x-1)^2(x+4)$ by $x - 1$.

29. $(x+1)(x^2-4)$ by $x + 2$.

30. $(x^2 - a^2)(x + a)$ by $x^2 + 2ax + a^2$.

31. $(x+y)^2 - z^2$ by $x + y - z$.

32. $x^2 - 2xy + y^2 - z^2$ by $x - y + z$.

33. $x^4 + 64$ by $x^2 + 4x + 8$.

34. $x^4 + x^2 + 1$ by $x^2 - x + 1$.

35. $x^4 + 9x^2 + 81$ by $x^2 - 3x + 9$.

36. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ by $x + y + z$.

37. $ax + ay + bx + by$ by $x + y$.

38. $ac - bc - ad + bd$ by $a - b$.

39. $x^{12} + y^{12}$ by $x^4 + y^4$.

40. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

115. MISCELLANEOUS EXAMPLES.

Factor the following examples:

1. $x^2 + 73x + 72$.

2. $x^2 - 14ax + 45a^2$.

3. $12x^4 - 60x^2 - 288$.

4. $ax^2 - 4a^2x + 4a^3$.

   Ans. $a, x - 2a, x - 2a$.

5. $2x^3y - 3xy^2 - 4xy$.

6. $x^2 - (a - b)x - ab$.

7. $(a-b)^2 - 11(a-b) + 18$.

8. $3x^2y^2 + 26axy + 35a^2$.

9. $x^3 - a^3$.

10. $x^{15} - x^9$.

11. $x^{15} - x^7$.

12. $2 - 50a^2$.

13. $18x^2 - 32y^2$.

14. $(x+1)^2 - x^2$.

15. $4(x-y)^3 - (x-y)$.

16. $2ab - 2cd - c^2 + a^2 - d^2 + b^2$.

Ans. $a + b + c + d$, and $a + b - c - d$.

17. $a^3 + b^3 + a + b$.

18. $a^2 - b^2 + a - b$.

19. $a^2 - y^2 - 2yz - z^2$.

20. $1 - (x - y)^3$.

21. $x^2 - 4y^2 + x - 2y$.

22. $5a^4b^4 - 5ab$.

23. $a^2 - 9b^2 + a + 3b$.

24. $1 - (m^2 + n^2) + 2mn$.

25. $x^4y - x^2y^3 - x^3y^2 + xy^4$.

Ans. xy, $x + y$, and $(x - y)^2$.

26. $a^4 + b^4 - c^4 - d^4 + 2a^2b^2 - 2c^2d^2$.

27. $a^2 + x^2 - (y^2 + z^2) - 2(yz - ax)$.

28. $4 + 4x + 2ay + x^2 - a^2 - y^2$.

29. $21x^2 + 82x - 39$. Ans. $3x + 13$, and $7x - 3$.

30. $a^5 - 8a^2b^3$.

31. $(a + b)^4 - 1$.

32. $250(a - b)^3 + 2$.

33. $6x^2 - x - 77$.

34. $c^5d^3 - c^2 - a^2c^3d^3 + a^2$.

35. $x^6 - 4096$.

36. $75a^4 - 48b^4$.

37. $3x^5 + 96$.

38. $x^4 - (a^2 + b^2)x^2 + a^2b^2$.

39. $x^{16} + x^8 + 1$.

40. $x^4 + 4x^2 + 16$.

41. $x^4 + 25x^2 + 625$.

42. $x^{2n} + 16x^n + 48$.

43. $.0001x^4 - 1$.

44. $3x^2 + x - 2$.

45. $4x^4 - x^2 - 2x - 1$.

46. $x^3a^2 - 8y^3a^2 - 4x^3a^2 + 32y^3a^2$.

Ans. 3, a^2, $2y - x$, and $4y^2 + 2xy + x^2$.

47. $1 - x - x^3 + x^4$.

48. $x^2 + \left(a + \frac{1}{a}\right)x + 1$.

49. $x^4 - 4y^4 + x^2 + 2y^2$.

50. $3x^2 - 3y^2 + 6x + 6y$.

51. $a^9 - 64a^3 - a^6 + 64$.

52. $216a^3 - \frac{b^3}{8}$.

53. $x^4 - 15x^2y^2 + 9y^4$.

54. $\frac{1}{8}a^3 + \frac{1}{27}b^6$.

55. $x^3 - 9x^2 + 9x - 1$. Ans. $x - 1$, and $x^2 - 8x + 1$.

56. $a^2x^2 + 2a^2bx + a^2b^2 - a^4b^4$.

57. $x^6 - x^4 + 1 - x^2$. Ans. $x^2 + 1$, $(x + 1)^2$, and $(x - 1)^2$.

58. $a^4 + a^2b^2 - b^2c^2 - c^4$. Ans. $a^2 - c^2$, and $a^2 + b^2 + c^2$.

59. $(x^2 + x - 20)(x^2 - x - 30)$.

60. $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5$.

61. $xy(1 + z^2) + z(x^2 + y^2)$. Ans. $x + yz$, and $y + xz$.

62. $32a^5 - 243b^{10}$.

63. $12x^2 + 14x + 4$.

64. $8y^2 + 18y + 7$.

65. $3x^2 + 3x - 18$.

66. $6x^2 + 7x - 5$.

67. $2x^2 - 3x + 1$.

68. $2x^3 - x^2 + 8x - 4$.

69. $5x^6 + 15x^5 - 5x^3 - 15x^2$.

70. $a^2 - b^2 - c^2 + d^2 - 2(ad - bc)$.

71. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.

72. $a^4 - 2a^2bc - b^4 - b^2c^2 - c^4$.
Ans. $a^2 - bc + b^2 + c^2$, and $a^2 - bc - b^2 - c^2$.

73. $(x^3 - 3x^2)^2 - (3x - 6)^2$.

74. $(a + b)^2 + (c + d)^2 + 2(ac + ad + bc + bd)$.

75. $a^2 + b^2 + c^2 - 2ac - 2ab + 2bc$.

76. $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2$.

77. $a^3 + b^3 + c^3 - 3abc$.

78. $a^3 + b^3 + 1 - 3ab$.

79. $4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$.

80. $x^3 - 3x^2y + 3xy^2 - y^3$.

81. $x^3 + 3x^2 + 3x + 1$.

CHAPTER VIII.

GREATEST COMMON DIVISOR.

116. A **Common Divisor** of two or more algebraic expressions is an expression which will divide each of them without remainder.

117. The **Greatest Common Divisor** of two or more algebraic expressions is the expression of the *highest degree* which will divide each of them without remainder.

118. From this definition it is evident that the greatest common divisor of two or more algebraic expressions must contain all the factors common to the expressions, and no others.

NOTE. The names *greatest common measure, highest common measure, highest common divisor, highest common factor,* are used by different authors to mean the same thing as *greatest common divisor.*

CASE I.

119. To find the Greatest Common Divisor of Monomials and Polynomials which can be resolved into Factors by Inspection.

1. Find the greatest common divisor of $12\,a^3 b^4 c^4$, $36\,a^2 b^3$, and $8\,a^5 b c^2$.

$$\begin{aligned} 12\,a^3 b^4 c^4 &= 2^2 \cdot 3\,a^3 b^4 c^4 \\ 36\,a^2 b^3 &= 2^2 \cdot 3^2\,a^2 b^3 \\ 8\,a^5 b c^2 &= 2^3 \cdot a^5 b c^2 \\ \hline \therefore \text{G. C. D.} &= 2^2 \cdot a^2 b \end{aligned}$$

It is evident that the highest power of 2 which will divide all three expressions is 2^2; of a, a^2; of b, b; and that c will not divide them all, therefore the greatest common divisor is $2^2 a^2 b$.

2. Find the greatest common divisor of $x^2 - y^2$, $x^2 + 2xy + y^2$, and $x^2 + xy$.

$$\begin{aligned} x^2 - y^2 &= (x + y)(x - y) \\ x^2 + 2xy + y^2 &= (x + y)^2 \\ x^2 + xy &= x(x + y) \\ \hline \therefore \text{G. C. D.} &= x + y \end{aligned}$$

From these examples we derive the following

Rule.

Separate each expression into its prime factors; then take every factor common to the given expressions the least number of times it occurs in any one of them for the greatest common divisor required.

Find the greatest common divisor of:

3. $9x^2y^4z^3$, $12xy^3z$.
4. $17a^2b^2c^2$, $34abc^3$, $51ab^2c$.
5. $24a^3b^4c^2$; $16a^3b^4c^3$, $40a^2b^4c^5$.
6. $25x^2yz^2$, $100x^3y^3z^3$; $125xy^2$.
7. $x^2 + xy$, $x^2 - y^2$.
8. $(x + y)^2$, $x^2 - y^2$.
9. $2x^2 - 2xy$, $x^3 - x^2y$.
10. $6x^2 - 9xy$, $4x^2 - 9y^2$.
11. $a^3 - a^2x$, $a^3 - ax^2$, $a^4 - ax^3$.
12. $a^2 - x^2$, $a^2 - ax$, $a^2x - ax^2$.
13. $x^2 + x$, $(x + 1)^2$, $x^3 + 1$.
14. $6(a + b)^5$, $15(a + b)^3$.
15. $24(a^2 - 9)$, $16(a - 3)^2$.
16. $x^2 - 1$, $x^3 - 1$, $(x - 1)^2$.
17. $a^2 - b^2$, $(a - b)^2$, $(a + b)^2$, $a^4 - b^4$.

18. $30x - 6,\ 100x^2 - 4.$
19. $x^2 - 2xy + y^2,\ (x - y)^3.$
20. $4x^2 - 1,\ (2x + 1)^2.$
21. $x^2 - 5x + 4,\ x^2 + x - 2.$
22. $x^2 - 18x + 45,\ 2(x^2 - 9).$
23. $x^2 - x - 20,\ x^2 - 9x + 20.$
24. $2x^2 - x - 1,\ 3x^2 - x - 2.$
25. $x^3 + 8y^3,\ x^2 + xy - 2y^2.$
26. $c^2x^2 - d^2,\ acx^2 - bcx + adx - bd.$
27. $x^2 + (a + b)x + ab,\ x^2 + (a + c)x + ac.$
28. $x^2 - (a - c) - ac,\ x^2 - (a + c)x + ac.$
29. $x^2 + (3 + y)x + 3y,\ x^2 + 5x + 6.$
30. $ab(x + a),\ a(x^2 + ax - bx - ab),\ b(x^2 + ax).$
31. $15(x^6 - y^6),\ 6(x + y)(x^5 - y^5).$
32. $2x^2 + 9x + 4,\ 2x^2 + 11x + 5,\ 2x^2 - 3x - 2.$
33. $ax^2 + 2a^2x + a^3,\ 2ax^2 - 4a^2x - 6a^3,\ 3(ax + a^2)^2.$
34. $3a^2 + 9ab,\ a^3 - 9ab^2,\ a^3 + 6a^2b + 9ab^2.$
35. $x^3 - 125,\ x^2 - 25,\ x^2 - 10x + 25.$

Case II.*

120. To find the Greatest Common Divisor of Polynomials which cannot be factored by Inspection.

To deduce a rule for finding the greatest common divisor of two or more numbers, we demonstrate the two following theorems:

THEOREM I.

121. *A common divisor of two polynomials is also a divisor of the sum or the difference of any multiples of each.*

* See Preface.

Let A and B be two polynomials, and let d be their common divisor; d is also a divisor of $mA \pm nB$.

Suppose $\quad A \div d = p$; i.e. $A = dp$, and $mA = dmp$,

and $\quad B \div d = q$; i.e. $B = dq$, and $nB = dnq$;

then $\quad mA \pm nB = dmp \pm dnq = d(mp \pm nq)$.

That is, d is contained in $mA + nB$, $mp + nq$ times, and in $mA - nB$, $mp - nq$ times; that is, d is a divisor of the sum or the difference of any multiples of A and B.

THEOREM II.

122. *The greatest common divisor of two polynomials is also the greatest common divisor of the less and the remainder after dividing the greater by the less.*

Let A and B be two expressions, and A not lower in degree than B, and let the process of dividing be as appears in the margin. Then, as the dividend is equal to the product of the divisor by the quotient plus the remainder,

$$\begin{array}{l} B)\ A\ (q \\ \quad \underline{qB} \\ \quad\ \ r \end{array}$$

$$A = r + qB. \qquad (1)$$

And, as the remainder is equal to the dividend minus the product of the divisor by the quotient,

$$r = A - qB. \qquad (2)$$

Therefore, according to the preceding theorem, from (1) any divisor of r and B must be a divisor of A; and from (2) any divisor of A and B, a divisor of r; that is, the divisors of A and B, and B and r, are identical, and therefore the greatest common divisor of A and B must also be the greatest common divisor of B and r.

In the same way, the greatest common divisor of B and r is the greatest common divisor of r and the remainder after dividing B by r.

123. Hence, to find the greatest common divisor of any two polynomials, we have the following

Rule.

After removing every monomial factor possible from each expression, arrange the resulting expressions, according to the

descending powers of some common letter, and divide the expression which is of the higher degree by the other. Continue the division until the remainder is of a lower degree than the divisor. Then make the remainder a new divisor and the divisor a new dividend; and continue the process until there is no remainder. The last divisor, together with the common monomial factors, removed at the beginning of the operation, will be the greatest common divisor.

NOTE. Monomial factors should be rejected when possible, and introduced only to avoid fractions. If, after the removal of such factors at any point of the process, polynomials of the same degree appear, either may be used as the divisor, though it is better to take as the divisor the one whose first term has the smaller coefficient.

1. Find the greatest common divisor of $4x^3-3x^2-24x-9$ and $8x^3-2x^2-53x-39$.

$$\begin{array}{l}
4x^3-3x^2-24x-9)\,8x^3-2x^2-53x-39\,(2 \\
\qquad\qquad\qquad\quad \underline{8x^3-6x^2-48x-18} \\
\qquad\qquad\qquad\qquad 4x^2-5x-21)\,4x^3-3x^2-24x-9\,(x \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{4x^3-5x^2-21x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2x^2-3x-9)\,4x^2-5x-21\,(2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{4x^2-6x-18} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x-3)\,2x^2-3x-9\,(2x+3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2x^2-6x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3x-9 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3x-9
\end{array}$$

$\therefore$ G. C. D. $= x-3$.

The following arrangement saves rewriting the divisor when it becomes the dividend:

$$\begin{array}{r|r|r|l}
x & 4x^3-3x^2-24x-9 & 8x^3-2x^2-53x-39 & 2 \\
 & \underline{4x^3-5x^2-21x} & \underline{8x^3-6x^2-48x-18} & \\
2x & 2x^2-3x-9 & 4x^2-5x-21 & 2 \\
 & \underline{2x^2-6x} & \underline{4x^2-6x-18} & \\
3 & 3x-9 & x-3 & \\
 & \underline{3x-9} & &
\end{array}$$

$\therefore$ G. C. D. $= x-3$.

2. Find the greatest common divisor of $10x^3+35x^2+30x$ and $4x^4+26x^3+58x^2+42x$.

$$\begin{array}{r|r|r|l}
5x & 10x^3+35x^2+30 & 2x\,|\,4x^4+26x^3+58x^2+42x & \\
\hline
x & 2x^2+7x+6 & 2x^3+13x^2+29x+21 & x\\
 & 2x^2+3x & 2x^3+7x^2+6x & \\
\hline
2 & 4x+6 & 6x^2+23x+21 & 3\\
 & 4x+6 & 6x^2+21x+18 & \\
\hline
 & & 2x+3 &
\end{array}$$

$\therefore$ G. C. D. $= x(2x+3)$.

3. Find the greatest common divisor of $3x^2-2x-1$ and $3x^3-9x+6$.

$$\begin{array}{r|r|r|l}
3 & 3x^2-2x-1 & 3x^3-9x+6 & x\\
 & 3x^2-12x+9 & 3x^3-2x^2-x & \\
\hline
 & 10\,|\,10x-10 & 2\,|\,2x^2-8x+6 & \\
 & x-1 & x^2-4x+3 & x\\
 & & x^2-x & \\
\hline
 & & -3x+3 & -3\\
 & & -3x+3 & \\
\hline
\end{array}$$

$\therefore$ G. C. D $= x-1$.

4. Find the greatest common divisor of $20x^2+12x-11$ and $6x^3+x^2-1$.

$$\begin{array}{r|r|r|l}
10x & 20x^2+12x-11 & 6x^3+\quad x^2-1 & \\
 & 20x^2-10x & 10 & \\
\hline
11 & 22x-11 & 60x^3+10x^2-10 & 3x\\
 & 22x-11 & 60x^3+36x^2-33x & \\
\hline
 & & -26x^2+33x-10 & \\
 & & 10 & \\
\hline
 & & -260x^2+330x-100 & -13\\
 & & -260x^2-156x+143 & \\
\hline
 & & 243\,|\,486x-243 & \\
 & & 2x-1 &
\end{array}$$

$\therefore$ G. C. D. $= 2x-1$.

Find the greatest common divisor of:

5. $6x^2 + 7x - 3$, $12x^2 + 16x - 3$.

6. $x^2 + 8x + 15$, $x^2 + 9x + 20$.

7. $x^2 + 12x + 35$, $x^2 + 13x + 42$.

8. $x^3 + 3x^2 + 4x + 4$, $x^3 + 4x^2 + 5x + 6$.

9. $2x^2 + 18x + 36$, $2x^3 + 20x^2 + 48x$.

10. $3x^4 - x - 3x^3 - 2x^2 - 1$, $-3x^3 - x + 6x^4 - x^2 - 1$.
Ans. $3x^2 + 1$.

11. $2x^5 - 2x^4 + x^3 + 3x^2 - 6x$, $8x^5 - 4x^4 + 6x^2 - 18x$.

12. $2a^2x - 2a^3 + 3x^3 - 3ax^2$, $12ax^2 + 8a^3 + 3x^3 + 2a^2x$.

13. $2x^2 - 14x + 20$, $20x - 25x^2 + 4x^3 + 25$.

14. $12ax - 9a + a^3 - 4ax^2$, $2a^2 - 4ax^2 + a^3 + 8ax - 3a$.

15. $2x^4 + 9x^3 + 14x + 3$, $3x^4 + 14x^3 + 9x + 2$.
Ans. $x^2 + 5x + 1$.

16. $x^5 + 4x^3 + 16x$, $2x^5 - 8x - x^4 + 16x^2$.

17. $5x^3 + 2x - 7x^2 + 3x^4 + 2$, $5 + 12x + 2x^4 + 3x^3 - 2x^2$.

18. $y^4 - 2y^3 + 3y^2 - 6y$, $y^5 - y^3 - y^4 - 2y^2$.

19. $24x^4y + 72x^3y^2 - 6x^2y^3 - 90xy^4$, $6x^4y^2 + 13x^3y^3 - 4x^2y^4 - 15xy^5$.
Ans. $xy(2x^2 + xy - 3y^2)$.

20. $54x^2a^5 + 4x^5a^2 + 10x^4a^3 - 60x^3a^4$, $24x^5a^3 + 30x^3a^5 - 126x^2a^6$.

21. $4x^5 + 14x^4 + 20x^3 + 70x^2$, $56x^3 + 8x^7 + 28x^6 - 8x^5 - 12x^4$.

22. $72a^2x - 420a^3 + 72x^3 - 12ax^2$, $42ax^2 + 18x^3 - 282a^2x + 270a^3$.
Ans. $6(3x - 5a)$.

23. $7x^4 - 10ax^3 + 3a^2x^2 - 4a^3x + 4a^4$, $8x^4 - 13ax^3 + 5a^2x^2 - 3a^3x + 3a^4$.

24. $8x^5 - 55x^3 + 3$, $3x^5 - 55x^2 + 8$.

25. $4x^5 + 14ax^4 - 18a^3x^2$, $24ax^3 + 30a^3x + 126a^4$.

26. $2x^2(x^4 - 2x^3 - 7x^2 + 16x + 7)$, $6x(x^3 - 6x^2 - 26x - 9)$.
Ans. $2x(x^2 + 3x + 1)$.

27. $2x^6 - 2ax^5 - 16a^4x^2$, $3ax^5 - 3a^2x^4 + 6a^3x^3 - 9a^4x^2 - 30a^5x$.

28. $21y^3 - 14ay^2 + 21a^2y - 14a^3$, $54y^3z - 16a^3z$.

29. $6x^4 - 13x^3 + 3x^2 + 2x$, $6x^4 - 9x^3 + 15x^2 - 27x - 9$.

30. $4ax^4 + 2ax^3 - 16ax^2 - 2ax + 12a$, $4x^4 + 12x^3 - x^2 - 27x - 18$.

124. To find the Greatest Common Divisor of three or more Expressions.

Find the greatest common divisor of two of them, then of this result and a third, and so on; the last divisor will be the greatest common divisor sought.

Find the greatest common divisor of:

1. $a^2 + 3ab + 2b^2$, $a^2 + ab - 2b^2$, $a^3 + 2a^2b - ab^2 - 2b^3$.

2. $39x^2 - 178x + 39$, $39x^2 - 184x + 65$, $27x^2 - 114x - 13$.

3. $x^3 - 6x^2 + 11x - 6$, $x^3 - 9x^2 + 26x - 24$, $x^3 - 8x^2 + 19x - 12$.
Ans. $(x - 3)$.

4. $x^3 - 9x^2 + 26x - 24$, $x^3 - 10x^2 + 31x - 30$, $x^3 - 11x^2 + 38x - 40$.

5. $x^3 - 3x^2 + 3x - 1$, $x^3 - x^2 - x + 1$, $x^4 - 2x^3 + 2x - 1$, $x^4 - 2x^3 + 2x^2 - 2x + 1$.

CHAPTER IX.

LEAST COMMON MULTIPLE.

125. A **Common Multiple** of two or more algebraic expressions is an expression that can be divided by each of them without remainder.

126. The **Least Common Multiple** of two or more algebraic expressions is the expression of the *lowest degree* that can be divided by each of them without remainder.

127. It is evident from the above definitions that a common multiple of two or more expressions must contain the factors of these expressions; and the *least* common multiple of two or more expressions must contain *only* the factors of these expressions.

CASE I.

128. To find the Least Common Multiple of Monomials, and Polynomials which can be resolved into Factors by Inspection.

1. Find the least common multiple of $8\,a^2b^2$, $24\,a^4b^2c^2$, and $4\,a\,b\,c^3$.

$$\begin{aligned} 8\,a^2b^2 &= 2^3\quad a^2b^2 \\ 24\,a^4b^2c^2 &= 2^3\cdot 3\,a^4b^2c^2 \\ 4\,a\,b\,c^3 &= 2^2\quad a\,b\,c^3 \end{aligned}$$

$$\therefore\ \text{L. C. M.} = 2^3\cdot 3\,a^4b^2c^3 = 24\,a^4b^2c^3$$

It is evident that no number which contains a power of 2 less than 2^3, of a less than a^4, of b less than b^2, of c less than c^3, and which does not contain 3, can be divided by each of these numbers; therefore the least common multiple is $2^3\cdot 3\,a^4b^2c$.

2. Find the least common multiple of $4\,(x^2-1)$, $6\,(x-1^2)$, $8\,(x^2+2\,x+1)$.

$$\begin{aligned} 4\,(x^2-1) &= 2^2 \quad (x+1)\,(x-1) \\ 6\,(x-1)^2 &= 3\cdot 2\,(x-1)^2 \\ 8\,(x^2+2\,x+1) &= 2^3 \quad (x+1)^2 \\ \hline \therefore\ \text{L. C. M.} &= 2^3\cdot 3\,(x+1)^2\,(x-1)^2 \end{aligned}$$

From these examples we derive the following

Rule.

Separate each expression into its prime factors, and then take every factor the greatest number of times it occurs in any one of the expressions for the least common multiple required.

Find the least common multiple of:

3. $8\,a^2b^2,\ 24\,a^4b^2c^2,\ 18\,a\,b\,c^3.$

4. $15\,a^3b^4,\ 20\,a^2b^2c^2,\ 30\,a\,c^3.$

5. $35\,x^2y^3z,\ 42\,x^3yz^2,\ 30\,x\,y^2z^3.$

6. $21\,x^3,\ 7\,x^2\,(x+1).$

7. $x^2-1,\ x^2+x.$

8. $6\,x^2-2\,x,\ 9\,x^2-3\,x.$

9. $4\,x^2y-y,\ 2\,x^2+x.$

10. $x-2,\ (x-2)^2.$

11. $x-y,\ x+y,\ x^2-y^2.$

12. $1-3x,\ 1+3x,\ 1-3x^2.$

13. $x-2,\ x+2,\ x^2+4.$

14. $6\,(x+y)^3,\ 9\,(x+y)^5.$

15. $4\,(x+1)^2,\ 6\,(x^2-1).$

16. $3\,(a-1),\ 2\,(a-1)^2,\ (a-1)^3.$

17. $c\,(a^2-c^2),\ (a+c)\,c,\ c.$

18. $8\,(1-x),\ 8\,(1+x),\ 4\,(1+x^2).$

19. $a+b,\ a-b,\ a^2+b^2,\ a^4+b^4.$

20. $4\,a^3\,(a+x),\ 4\,a^3\,(a-x),\ 2\,a^2\,(a^2-x^2).$

21. $3\,a+1,\ 9\,a^2-1,\ 27\,a^3+1.$

22. $a^3+a^2b,\ a\,(c-b),\ a^2-b^2.$

23. $2\,a\,x\,y\,(x-y),\ 3\,a\,x^2\,(x^2-y^2),\ 4\,y^2\,(x+y)^2.$

24. $6(x^2-9)$, $9(x+3)$, $15(x-4)$, $10(x^2-x-12)$.

25. $4a+4b$, $6a^2-24b^2$, $a^2-3ab+2b^2$.

26. $2a-3$, $6a+9$, $3(4a^2-9)$.

27. $9-a^2$, $a+3$, $3-a$.

28. x^3+y^3, x^3y-y^4, x^6-y^6.

29. x^2-y^2, x^2-xy+y^2, x^2+xy+y^2.

30. $4(a^2b-ab^2)$, $8(a^3-ab^2)$, $12(ab^2+b^3)$.

31. x^3-y^3, x^3+y^3, xy^2-y^3, x^3-xy^2.

32. x^2-5x+6, x^2-6x+9.

33. x^2+x-2, x^2-2x+1.

34. $2x^2-7x+3$, $2x^2+5x-3$.

35. $3x^2+7x+2$, x^2-x-6.

36. $x^2-(a+b)x+ab$, $x^2-(a-b)x-ab$.

37. $(x-y)^2-(a-b)^2$, $(x-a)^2-(y-b)^2$.

38. $(x+2)^2-(x-2)^2$, x^4.

39. $(a+b)^2-c^2$, $(a+b+c)^2$.

40. $3(x-2)$, $7(x-3)(x-2)$.

41. $(3x-2)(2x-5)$, $(2x-5)(x+7)$, $(x+7)(x-1)$.

42. $(a+b)^3$, $(a+b)(a^2-ab+b^2)$, $ax(a^2-ab+b^2)$.

43. $(a-b)(a-c)$, $(b-c)(b-a)$, $(c-a)(c-b)$.

44. $a(a^2-b^2)(a^2-c^2)$, $b(b^2-c^2)(b^2-a^2)$, $c(c^2-a^2)(c^2-b^2)$.

45. $1-x$, $(1-x^2)^2$, $(1+x)^3$.

46. $(bc^2-abc)^2$, $b^2(ac^2-a^3)$, $a^2c^2+2ac^3+c^4$.

47. $1+x+x^2$, $1+x^2+x^4$, $1-x+x^2$.

48. $x^{2n}-y^{2n}$, $(x^n-y^n)^2$.

49. $x^{2n}+2x^ny^n+y^{2n}$, $x^{2n}-y^{2n}$, $x^{2n}-2x^{2n}y^{2n}+y^{2n}$.

50. $21x(xy-y^2)^2$, $35(x^4y^2-x^2y^4)$, $15y(x^2+xy)^2$.

Case II.

129. To find the Least Common Multiple of Polynomials which cannot be readily factored by Inspection.

Let A and B stand for any two algebraic expressions, and let G stand for their greatest common divisor, and L for their least common multiple.

Let the quotients of A and B by G be represented, respectively, by q and q'.

Since G is the greatest common divisor of A and B, q and q' can have no common factor; therefore the least common multiple of A and B must be $G \cdot q \cdot q'$; that is, $L = G \cdot q \cdot q'$, or

$$L = G \cdot q \cdot \frac{G q'}{G} = A \cdot \frac{B}{G} = \frac{A}{G} \cdot B$$

130. Hence, to find the least common multiple of two polynomials which cannot be readily factored by inspection, or, indeed, of any two algebraic expressions, we have the following

Rule.

Divide one of the expressions by their greatest common divisor, and multiply this quotient by the other expression for the least common multiple required.

1. Find the least common multiple of $6x^2 + 7x + 2$, $24x^2 + 54x + 21$.

$$\begin{array}{r|r|r|l} 3x & 6x^2+7x+2 & 24x^2+54x+21 & 4 \\ & 6x^2+3x & 24x^2+28x+8 & \\ \hline 2 & 4x+2 & 13 \,|\, 26x+13 & \\ & 4x+2 & \text{G.C.D.} = 2x+1 & \end{array}$$

$$\therefore \text{ L. C. M.} = \frac{(6x^2+7x+2)(24x^2+54x+21)}{2x+1}$$
$$= 3(2x+1)(3x+2)(4x+7)$$

Find the least common multiple of:

2. $2x^2 + 3x - 20,\ 6x^3 - 25x^2 + 21x + 10.$

3. $x^2 - 15x + 36,\ x^3 - 3x^2 - 2x + 6.$

4. $x^3 - x^2 - 7x + 15,\ x^3 + x^2 - 3x + 9.$

5. $a^6 - 6a^4 + 9a^2 - 4,\ a^6 + a^5 - 2a^4 + 3a^2 - a - 2.$
Ans. $(a^2 - 1)^2 (a^2 - 4)(a^3 - a + 2).$

6. $x^3 - x^2 + x + 3,\ x^4 + x^3 - 3x^2 - x + 2.$

7. $x^4 - 5x^3 + 20x - 16,\ x^4 - 2x^3 - 3x^2 + 8x - 4.$

8. $a^3 + 6a^2 + 11a + 6,\ a^3 + 10a^2 + 29a + 20.$
Ans. $(a + 1)(a + 2)(a + 3)(a + 4)(a + 5).$

131. To find the Least Common Multiple of three or more Expressions.

Find the least common multiple of two of them, then of this result and a third, and so on; the last common multiple will be the least common multiple sought.

Find the least common multiple of:

1. $3x^3 - 7x^2y + 5xy^2 - y^3,\ \ x^2y + 3xy^2 - 3x^3 - y^3,\ 3x^3 + 5x^2y + xy^2 - y^3.$

2. $x^2 - 3ax + 2a^2,\ 3x^2 - 19ax + 28a^2,\ x^2 - 5ax + 4a^2.$
Ans. $3x^4 - 28ax^3 + 91a^2x^2 - 122a^3x + 56a^4.$

3. $x^3 + 3x^2 - 6x - 8,\ x^3 - 2x^2 - x + 2,\ x^2 + x - 6.$

4. $2x^4 + x^3 - 8x^2 - x + 6,\ 4x^4 + 12x^3 - x^2 - 27x - 18,\ 4x^5 + 8x^4 - 13x^3 - 26x^2 + 9x + 18.$
Ans. $4x^5 + 8x^4 - 13x^3 - 26x^2 + 9x + 18.$

5. $x^3 - 3x^2 + 3x - 1,\ \ x^3 - x^2 - x + 1,\ \ x^4 - 2x^3 + 2x - 1,\ \ x^4 - 2x^3 + 2x^2 - 2x + 1.$

CHAPTER X.

FRACTIONS.

132. WHEN division is expressed by writing the dividend over the divisor with a line between, the expression is called a **Fraction**. As a fraction, the dividend is called the numerator, and the divisor the denominator.

Thus $\frac{a}{b}$ means $a \div b$. Hence it follows that $\frac{a}{b} \times b = a$. That is, the algebraic fraction $\frac{a}{b}$, *where a and b are supposed to have any values whatever*, is that expression which, when multiplied by b, becomes a.

133. The words *fractional* and *integral*, as applied to algebraic expressions, always apply to the algebraic *form* of the expression, and not to its numerical value for particular numerical values of the letters involved.

134. The **Terms** of a fraction are its numerator and denominator. These **Terms** may be polynomials, and hence each be composed of several *terms* (§ 14). Their dividing line also performs the office of a vinculum, and any removal of the line and sign preceding it calls for a corresponding change of the signs of the terms of the numerator, in accordance with the laws which govern the omission of brackets.

135. A **Simple Fraction** is one whose terms are integral in form; as, $\frac{a}{b}$, $\frac{a+c}{b-d}$.

136. A **Mixed Expression, or Mixed Number,** is an expression composed of both integral and fractional forms; as,

$$a + \frac{b}{c},\ a - b - \frac{c-d}{e+f}.$$

137. From the principles of division, and § 134, it follows that

$$\frac{ab}{a} = \frac{-ab}{-a} = -\frac{-ab}{a} = -\frac{ab}{-a} = +b;$$

and that $$\frac{-ab}{a} = \frac{ab}{-a} = -\frac{ab}{a} = -b;$$ that is,

(*a*) The value of a fraction *is not changed,*

(1) *If the sign of every term of the numerator and denominator is changed.*

(2) *If the sign of every term of the numerator and the sign before the fraction are changed.*

(3) *If the sign of every term of the denominator and the sign before the fraction are changed.*

(*b*) But the value of a fraction *is changed,*

(4) *If every sign of the numerator, or denominator, or the sign before the fraction, is changed.*

138. From the rules of multiplication it follows that

$$\begin{aligned}
(a)\,(b)\,(c) &= +\,abc\\
(-a)\,(b)\,(c) &= -\,abc\\
(-a)\,(-b)\,(c) &= +\,abc\\
(-a)\,(-b)\,(-c) &= -\,abc\\
(a-b)\,(c-d) &= \,ac - ad - bc + bd\\
(b-a)\,(c-d) &= -\,ac + ad + bc - bd
\end{aligned}$$

and $$(b-a)\,(d-c) = \,ac - ad - bc + bd$$

Hence,

$$\frac{abc}{1} = \frac{(-a)\,(-b)\,(c)}{1} = -\frac{(-a)\,(b)\,(c)}{1} = -\frac{(-a)\,(-b)\,(-c)}{1}.$$

Also $$\frac{1}{(a-b)\,(c-d)} = \frac{1}{(b-a)\,(d-c)} = -\frac{1}{(b-a)\,(c-d)}$$

That is, *the value of a fraction is not changed, if the signs of two of its factors, or of an even number of its factors, in the numerator or denominator, or in both, are changed; but the value of a fraction is changed, if the sign of one of its factors, or of an odd number of its factors, in either the numerator or denominator is changed.*

139. EXERCISES.

Express the following fractions in four different ways:

1. $\frac{a-b}{c-d}=\frac{b-a}{d-c}=-\frac{b-a}{c-d}=-\frac{a-b}{d-c}$.

2. $\frac{x}{y-z}$.

3. $\frac{a-b}{c}$.

4. $\frac{a-b}{c-d-e}$.

5. $-\frac{x-y-z}{e-f}$.

6. $\frac{a-b+d}{e+f-g}$.

7. $\frac{a}{(a-b)(b-c)}$.

8. $\frac{a-b}{(b-c)(c-d)}$.

9. $\frac{(a-b)(c-d)}{(a-b)(c-d)(e-f)}$.

10. $\frac{a(c-d)}{(e-f)(b-c)}$.

Write the following fractions so that a and b shall become positive:

11. $\frac{-a}{x-b}=\frac{a}{b-x}$.

12. $-\frac{x-a}{-y-b}$.

13. $\frac{-a-x+y}{z-a}$.

14. $\frac{x-a}{y-b}$.

15. $\frac{x+y-a}{x-y-b}$.

16. $\frac{-a(b-c)}{d-b}$.

140. We are now to prove that $\frac{a}{b}=\frac{a\,m}{b\,m}$, and that $\frac{a\,m}{b\,m}=\frac{a}{b}$, for all values of a, b, and m.

$$\begin{aligned}\frac{a}{b} &= a \div b, \text{ by definition,}\\ &= a \div b \div m \times m, \text{ by axiom,}\\ &= a \times m \div b \div m, \text{ by law of commutation,}\\ &= (a\,m) \div (b\,m), \text{ by law of association,}\\ &= \frac{a\,m}{b\,m}.\end{aligned}$$

Conversely, $\frac{a\,m}{b\,m} = \frac{a}{b}$.

That is, *multiplying or dividing both numerator and denominator by the same number does not change the value of the fraction.*

141. REDUCTION OF FRACTIONS.

The *reduction* of fractions, that is, the changing of their form without changing their value, depends upon the preceding principles, and may be conveniently presented under the following cases.

CASE I.

142. To reduce a Fraction to its Lowest Terms.

A fraction is in its *lowest terms* when its numerator and denominator have no common factor.

Since the cancellation of the same factors from each term, or the dividing each term by the same factor, as, for instance, the greatest common factor, does not change the value of the fraction (§ 140), we have for the reduction of a fraction to its lowest terms the following

Rule.

Resolve the numerator and denominator of the fraction into their prime factors, and cancel all the common factors.

Or,

Divide both terms by their greatest common divisor.

1. Reduce $\frac{36\,a^4 b^2 c}{90\,a^5 b\,c^2}$ to its lowest terms.

$$\frac{36\,a^4 b^2 c}{90\,a^5 b\,c^2} = \frac{2^2 \cdot 3^2\,a^4 b^2 c}{2 \cdot 3^2 \cdot 5\,a^5 b\,c^2} = \frac{2\,b}{5\,a\,c}$$

2. Reduce $\frac{x^4 - x^2}{x^4 - 1}$ to its lowest terms.

$$\frac{x^4 - x^2}{x^4 - 1} = \frac{x^2\,(x^2 - 1)}{(x^2 + 1)\,(x^2 - 1)} = \frac{x^2}{x^2 + 1}$$

Reduce the following fractions to their lowest terms:

3. $\frac{14\,a\,b^2\,c^5}{21\,a^5\,b^2\,c}$.

4. $\frac{125\,x^7\,y^2\,z^{10}}{150\,x^2\,y^2\,z^2}$.

5. $\frac{2\,a\,b}{a^2 + a\,b}$.

6. $\frac{x^2\,y^2}{x^2 - x^2\,y^2}$.

7. $\frac{a^2 - b^2}{a^8 - b^8}$.

8. $\frac{(x - 1)^2}{x^2 - 1}$.

9. $\frac{x^2 - x^2\,y^2}{(x + x\,y)^2}$.

10. $\frac{x^4 + x^2}{x^4 - 1}$.

11. $\frac{a - b}{b^2 - a^2}$.

12. $\frac{a - 3}{9 - a^2}$.

13. $\frac{b^2 - a^2}{(a - b)^2}$.

14. $\frac{a^2\,x^2\,y^2 - x^4\,y^4}{x^4\,y^4 - a^4}$.

Ans. $\frac{-x^2\,y^2}{x^2\,y^2 + a^2}$.

15. $\frac{a^4 + 2\,a^2\,b^2 + b^4}{a^4 - b^4}$.

16. $\frac{x^6 - x^2}{x^8 - x^2}$.

17. $\frac{1 - 5\,a + 6\,a^2}{1 - 7\,a + 12\,a^2}$.

Ans. $\frac{1 - 2\,a}{1 - 4\,a}$.

18. $\frac{x^3\,y + 2\,x^2\,y + 4\,x\,y}{x^3 - 8}$.

19. $\frac{(a^3 - b^3)\,(-a\,b + a^2 + b^2)}{(a^3 + b^3)\,(a^2 + a\,b + b^2)}$.

20. $\frac{x^4 - 14\,x^2 - 51}{x^4 - 2\,x^2 - 15}$.

Ans. $\frac{x^2 - 17}{x^2 - 5}$.

21. $\frac{(a^6 - b^6)\,(a - b)}{(a^3 - b^3)\,(a^4 - b^4)}$.

22. $\dfrac{x(b-c)}{(a-b)(c-b)(a-c)}$.

25. $\dfrac{(a+b)^2-c^2}{(a+b+c)^2}$.

23. $\dfrac{a^2-ab+ax-bx}{a^2+ab+ax+bx}$.

26. $\dfrac{(x+y)^2-(a+b)^2}{(x-a)^2-(y-b)^2}$.

24. $\dfrac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$.

Ans. $\dfrac{x^2-ax+a^2}{x^2-a^2}$.

27. $\dfrac{(a+b+c)^2-(a-b-c)^2}{2a(b^2+2bc+c^2)}$.

Ans. $\dfrac{2}{b+c}$.

28. $\dfrac{abx^2+a^2xy+aby^2+b^2xy}{abx^2+a^2xy-aby^2-b^2xy}$.

29. $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}$.

30. $\dfrac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$. Ans. $\dfrac{a+b}{(c+b-a)(c-b+a)}$.

31. $\dfrac{x^3-23x+10}{5x^3-23x^2+4}$.

By § 123 the G. C. D. is found to be x^2-5x+2, and

$$\frac{(x^3-23x+10)\div(x^2-5x+2)}{(5x^3-23x^2+4)\div(x^2-5x+2)}=\frac{x+5}{5x+2}, \text{ Ans.}$$

32. $\dfrac{x^3-3x+2}{2x^3-3x^2+1}$.

33. $\dfrac{3x^2-8x+5}{x^3-4x^2+5x-2}$.

34. $\dfrac{x^3+5x^2+7x+3}{x^4+3x^3+4x^2+3x+1}$. Ans. $\dfrac{x+3}{x^2+x+1}$.

35. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$.

36. $\dfrac{2x^3+9x^2+7x-3}{3x^3+5x^2-15x+4}$. Ans. $\dfrac{2x+3}{3x-4}$.

37. $\dfrac{5a^5 + 10a^4x + 5a^3x^2}{a^3x + 2a^2x^2 + 2ax^3 + x^4}$.

38. $\dfrac{3a^2x^4 - 2ax^2 - 1}{4a^3x^6 - 2a^2x^4 - 3ax^2 + 1}$. Ans. $\dfrac{3ax^2 + 1}{4a^2x^4 + 2ax^2 - 1}$.

39. $\dfrac{x^4 + 2x^3 - 3x^2 - 7x - 2}{2x^4 + x^3 - 6x^2 - 5x - 1}$.

40. $\dfrac{ax^3 + 40a^4 - 5a^2x^2 - 99a^3x}{x^4 - 6ax^3 - 86a^2x^2 + 35a^3x}$. Ans. $\dfrac{a(x + 8a)}{x(x + 7a)}$.

41. $\dfrac{-x^2 - 48x + 12x^3 - 35}{19x^2 - 32x + 28x^3 - 15}$.

42. $\dfrac{7x^3 - 2x^2y - 63xy^2 + 18y^3}{5x^4 - 3x^3y - 43x^2y^2 + 27xy^3 - 18y^4}$.

Case II.

143. To reduce a Fraction to an Integral or Mixed Number.

1. Reduce $\dfrac{2x^2 - 7x - 1}{x - 3}$ to an integral or mixed number.

$$
\begin{array}{r|l}
x - 3 \,)\; 2x^2 - 7x - 1 & \left(2x - 1 + \dfrac{-4}{x-3}\right. \\
2x^2 - 6x \quad\quad\; & \text{or,} \\
\hline
-x - 1 & 2x - 1 - \dfrac{4}{x - 3} \quad (\S\ 134) \\
-x + 3 & \\
\hline
-4 &
\end{array}
$$

Since a fraction represents the quotient of the numerator divided by the denominator, we perform the indicated division, adding to the quotient the fraction formed by placing the remainder over the divisor. Hence,

To reduce a fraction to an integral or mixed number, we have the following

Rule.

Divide the numerator by the denominator, and if there is a remainder place it over the divisor, and add the fraction so formed to the quotient.

Reduce the following to integral or mixed numbers:

2. $\dfrac{a^2 - 2ab + c}{a}$.

3. $\dfrac{3x^2 + 9x + 2}{3x}$.

4. $\dfrac{2x^2 + ax - 3a^2}{x + a}$.

Ans. $2x - a - \dfrac{2a^2}{x + a}$.

5. $\dfrac{x^4 + 16}{x + 2}$.

6. $\dfrac{x^2 - 9y^2 + 13z^2}{x - 3y}$.

7. $\dfrac{x^3 - y^3}{x + y}$.

8. $\dfrac{x^5 - y^5}{x - y}$.

9. $\dfrac{a^3 - b^3}{(a - b)^2}$.

10. $\dfrac{x^2 - 3x + 1}{x - 2}$.

Ans. $x - 1 - \dfrac{1}{x - 2}$.

11. $\dfrac{a^4 + a^2 + 1}{a + 1}$.

12. $\dfrac{x^3 + 1}{x - 1}$.

13. $\dfrac{1 + x^2}{1 - x}$.

14. $\dfrac{x^3 - 3x^2a - 3xa^2 + a^3}{x - a}$.

15. $\dfrac{2x^4 - 6x^3 + 13x^2 - 15x + 8}{x^2 - x + 3}$.

16. $\dfrac{x^5 - x^3 - 2x^2 - 2x - 1}{x^2 - x - 1}$.

Ans. $x^3 + x^2 + x - \dfrac{x + 1}{x^2 - x - 1}$.

17. $\dfrac{x^4 + x^3 + x^2 + x + 1}{x^2 - x + 1}$.

Case III.

144. To reduce a Mixed Number to a Fractional Form.

It will be observed by referring to Case II., of which this is the converse, that the integral part of the mixed expression always stands for the quotient, the denominator of the fractional part for the divisor, and the numerator for the remainder; and that the dividing line also performs the office of a vinculum for the numerator. Hence,

To reduce a mixed number to a fractional form, we have the following

Rule.

Multiply the integral part by the denominator of the fraction; to the product add the numerator if the sign of the fraction is plus, and subtract it if the sign is minus, and under the result write the denominator.

1. Reduce $a - b + \dfrac{2b^2 - a^2}{a + b}$ to a fractional form.

$$(1) \quad a - b + \frac{2b^2 - a^2}{a + b} = \frac{a^2 - b^2 + 2b^2 - a^2}{a + b} = \frac{b^2}{a + b}$$

$$(2) \quad a - b - \frac{a^2 - 2b^2}{a + b} = \frac{a^2 - b^2 - a^2 + 2b^2}{a + b} = \frac{b^2}{a + b}$$

$$(3) \quad \frac{2b^2 - a^2}{a + b} + a - b = \frac{2b^2 - a^2 + a^2 - b^2}{a + b} = \frac{b^2}{a + b}$$

$$(4) \quad \frac{2b^2 - a^2}{a + b} - b + a = \frac{2b^2 - a^2 - b^2 + a^2}{a + b} = \frac{b^2}{a + b}$$

$$(5) \quad \frac{2b^2 - a^2}{a + b} - (b - a) = \frac{2b^2 - a^2 - (-a^2 + b^2)}{a + b} = \frac{b^2}{a + b}$$

(2), (3), (4), and (5) differ from (1) in form only. They are derived from (1) through §§ 137, 57, and 64.

2. Reduce $a + b + \dfrac{a^2 + b^2}{a - b}$ to a fractional form.

$$\begin{aligned} a + b + \frac{a^2 + b^2}{a - b} &= \frac{a^2 - b^2 + (a^2 + b^2)}{a - b} \\ &= \frac{a^2 - b^2 + a^2 + b^2}{a - b} \\ &= \frac{2a^2}{a - b} \end{aligned}$$

3. Reduce $a + b - \frac{a^2 + b^2}{a - b}$ to a fractional form.

$$a + b - \frac{a^2 + b^2}{a - b} = \frac{a^2 - b^2 - (a^2 + b^2)}{a - b}$$
$$= \frac{a^2 - b^2 - a^2 - b^2}{a - b}$$
$$= \frac{-2b^2}{a - b}, \text{ or } \frac{2b^2}{b - a}$$

Reduce to fractional form the following examples:

4. $a + \frac{b^2}{a}$.

5. $1 + \frac{2xy}{x^2 + y^2}$.

6. $1 - \frac{2xy}{x^2 + y^2}$.

7. $a - 2b - \frac{2b^2}{a - b}$.

8. $a - \frac{a(b - c)}{b}$.

9. $-\frac{x^2 + 1}{x + 1} + (x - 1)$.

10. $x^2 - xy + y^2 - \frac{2y^3}{x + y}$.

11. $x - a + y + \frac{a^2 - ay + y^2}{x + a}$. Ans. $\frac{x^2 + xy + y^2}{x + a}$.

12. $2x - 3 - \frac{3x - 1}{3x^2 - x + 2}$.

13. $\frac{a^2 - b^2}{2b} - (a - b)$.

14. $x^2 - 2 - \frac{x - 4}{x - 1}$.

15. $5a^2 - 4ab + 2b^2 + \frac{8a^3b + 11a^2b^2 - 12ab^3 + 4b^4}{2a^2 - 2b^2}$.

Ans. $\frac{10a^4 + 5a^2b^2 - 4ab^3}{2a^2 - 2b^2}$.

16. $\frac{x^4 + 2x^2 + 1}{x^2 + x + 1} - (x - x^2 - 1)$. Ans. $\frac{2x^4 + 3x^2 + 2}{x^2 + x + 1}$.

17. $a - b - \frac{(a + b)^2}{a - b}$.

18. $1 - \frac{a^2 + b^2 - c^2}{2ab}$.

19. $a^3 - b^3 + \frac{a^2b^2(a + b)}{a^2 - b^2}$.

20. $a^2 - \left(\frac{a^2 + b^2 - c^2}{2b}\right)^2$.

21. $1 + x + x^2 + x^3 + \frac{x^4}{1 - x}$. Ans. $\frac{1}{1 - x}$.

Case IV.

145. To reduce Fractions to their Least Common Denominator.

1. Reduce $\frac{a}{mx}$, $\frac{b}{my}$, and $\frac{c}{mz}$ to equivalent fractions having the least common denominator.

The required fractions must have for their denominators $mxyz$, the least common multiple of the given denominators.

If we multiply the numerators, a, b, c, by the quotients of $mxyz$ divided by mx, my, mz, respectively, we shall have, by § 140, the equivalent fractions required.

That is,
$$\frac{a}{mx} = \frac{a \times (yz)}{mx \times (yz)} = \frac{ayz}{mxyz}$$
$$\frac{b}{my} = \frac{b \times (xz)}{my \times (xz)} = \frac{bxz}{mxyz}$$
and
$$\frac{c}{mz} = \frac{c \times (xy)}{mz \times (xy)} = \frac{cxy}{mxyz}$$

Hence the following

Rule.

Find the least common multiple of the denominators for the least common denominator. For new numerators, multiply each numerator by the quotient arising from dividing this multiple by its denominator.

Note 1. Fractions should always first be reduced to their lowest terms.

Note 2. The familiar method of multiplying together the denominators for a new denominator, and each numerator by all the denominstors except its own, for a new numerator, is sometimes useful. In cases where the denominators are mutually prime, it is identical with the process above.

Note 3. Every integral form may be considered as a fraction with unity for its denominator; that is, $a = \frac{a}{1}$.

2. Reduce $\frac{a^2}{3xy}$, $\frac{b}{5x^2y}$, and $\frac{b^2}{15xy^3}$ to equivalent fractions having the least common denominator.

The L. C. D. is $15x^2y^3$.

$$\therefore \quad \frac{a^2}{3xy}, \quad \frac{b}{5x^2y}, \quad \text{and} \quad \frac{b^2}{15xy^3}$$

are equal, respectively, to $\frac{5a^2xy^2}{15x^2y^3}$, $\frac{3by^2}{15x^2y^3}$, and $\frac{b^2x}{15x^2y^3}$.

3. Reduce $\frac{1}{x(x-y)}$ and $\frac{1}{y(x+y)}$ to a common denominator.

The L. C. D. is $xy(x^2-y^2)$.

$$\therefore \quad \frac{1}{x(x-y)}, \quad \frac{1}{y(x+y)}$$

are equal, respectively, to $\frac{y(x+y)}{xy(x^2-y^2)}$, $\frac{x(x-y)}{xy(x^2-y^2)}$.

Reduce to equivalent fractions having their least common denominator:

4. $\frac{a+3}{5}$, $\frac{a+7}{10}$.

6. $\frac{1}{2x-3y}$, $\frac{x+y}{4x^2-9y^2}$.

5. $\frac{a-b}{ab}$, $\frac{b-c}{bc}$.

7. $\frac{4x^2}{3(a+b)}$, $\frac{xy}{6(a^2-b^2)}$.

Ans. $\frac{8x^2(a-b)}{6(a^2-b^2)}$, $\frac{xy}{6(a^2-b^2)}$.

8. $\frac{a+x}{a-x}$, $\frac{a-x}{a+x}$.

9. $\frac{1}{a-2x}$, $\frac{(a+2x)^2}{a^3-8x^3}$.

10. $\frac{2a^2-b^2}{a^2}$, $\frac{b^2-a^2}{b^2}$, $\frac{c^2-a^2}{c^2}$.

11. $\frac{1}{4a^3(a+x)}$, $\frac{1}{4a^3(a-x)}$, $\frac{1}{2a^2(a^2-x^2)}$.

Ans. $\frac{a-x}{4a^3(a^2-x^2)}$, $\frac{a+x}{4a^3(a^2-x^2)}$, $\frac{2a}{4a^3(a^2-x^2)}$.

12. $\frac{a}{(a-b)(b-c)}$, $\frac{b}{(a-b)(c-b)}$.

13. $\frac{3x}{1-x^2}$, $\frac{2}{x-1}$, $\frac{2}{1+x}$. 14. $\frac{x}{a+x}$, $\frac{a}{a-x}$, 1.

15. $\frac{a^2}{a^2-b^2}$, $\frac{b^2}{(a+b)^2}$, $\frac{ab}{(a-b)^2}$.

Ans. $\frac{a^2(a^2-b^2)}{(a^2-b^2)^2}$, $\frac{b^2(a-b)^2}{(a^2-b^2)^2}$, $\frac{ab(a+b)^2}{(a^2-b^2)^2}$.

16. $\frac{1}{x^2+5ax+6a^2}$, $\frac{2}{x^2+4ax+3a^2}$, $\frac{1}{x^2+3ax+2a^2}$.

17. $\frac{1}{4a^3(a+x)}$, $\frac{1}{4a^3(x-a)}$, $\frac{1}{2a^2(a^2+x^2)}$, $\frac{a^4}{a^8-x^8}$.

18. $\frac{1}{x}$, $\frac{1}{x-y}$, $\frac{1}{x^2+xy+y^2}$.

19. $\frac{6x}{1-x^2}$, $\frac{2x}{1+2x}$, $\frac{3x^2}{4x^2-1}$, $1\frac{1}{4}$.

146. ADDITION AND SUBTRACTION OF FRACTIONS.

1. Find the sum of $\frac{ab}{a}$ and $\frac{ac}{a}$, and also their difference.

From the distributive law of division it follows that

$$(ab+ac) \div a = ab \div a + ac \div a = b + a$$

Or, expressed in the fractional form,

$$\frac{ab+ac}{a} = \frac{ab}{a} + \frac{ac}{a} = b + a$$

Also, $$\frac{ab-ac}{a} = \frac{ab}{a} - \frac{ac}{a} = b - a$$

Conversely $$\frac{ab}{a} + \frac{ac}{a} = \frac{ab+ac}{a}$$

and $$\frac{ab}{a} - \frac{ac}{a} = \frac{ab-ac}{a}$$

Hence, for adding or subtracting fractions, we have the following

Rule.

Reduce the fractions, if necessary, to equivalent fractions having their least common denominator; then add or subtract their numerators, as the sign before the fraction directs, and write the result over the least common denominator.

2. Find the value of $\dfrac{3b+a}{5a}+\dfrac{7b-4a}{10a}$.

The L. C. D. is $10a$.

$$\begin{aligned}\therefore\quad &\frac{3b+a}{5a}+\frac{7b-4a}{10a}\\ &=\frac{2(3b+a)+(7b-4a)}{10a}\\ &=\frac{6b+2a+7b-4a}{10a}=\frac{13b-2a}{10a}\end{aligned}$$

3. Find the value of $\dfrac{x-2y}{xy}+\dfrac{3y-a}{ay}-\dfrac{3x-4a}{ax}$.

The L. C. D. is axy.

$$\begin{aligned}\therefore\quad &\frac{x-2y}{xy}+\frac{3y-a}{ay}-\frac{3x-4a}{ax}\\ &=\frac{a(x-2y)+x(3y-a)-y(3x-4a)}{axy}\\ &=\frac{ax-2ay+3xy-ax-3xy+4ay}{axy}\\ &=\frac{2ay}{axy}=\frac{2}{x}\end{aligned}$$

Simplify:

4. $\dfrac{x-1}{2}+\dfrac{x+3}{5}+\dfrac{x+7}{10}$.

5. $\dfrac{a-2b}{2a}-\dfrac{a-5b}{4a}+\dfrac{a+7b}{8a}$. Ans. $\dfrac{3(a+3b)}{8a}$.

6. $\dfrac{a-x}{x}+\dfrac{a+x}{a}-\dfrac{a^2-x^2}{2ax}$.

7. $\frac{2}{xy} - \frac{3y^2 - x^2}{xy^3} + \frac{xy + y^2}{x^2y^2}$. Ans. $\frac{x^3 + y^3}{x^2y^3}$.

8. $\frac{a-b}{ab} + \frac{b-c}{bc} + \frac{c-a}{ca}$.

9. $\frac{a^2 - bc}{bc} - \frac{ac - b^2}{ca} - \frac{ab - c^2}{ab}$.

10. $\frac{a}{x^n} + \frac{b}{x^{n-1}} + \frac{c}{x^{n-2}}$. Ans. $\frac{a + bx + cx^2}{x^n}$.

11. $\frac{a^2 + b^2}{c} - \frac{(a-b)^2}{c} + \frac{(a+b)^2}{c} + \frac{2a - 2b^2}{2c}$.

12. $\frac{a + 5b - 7c}{15} + \frac{3a - 7b + 5c}{20} - \frac{2a - b + 5c}{30}$.

13. $\frac{3(2a - 3b)}{8} - \frac{2(3a - 5b)}{3} + \frac{5(a-b)}{6}$.

14. $\frac{3a - 4b}{2} - \frac{2a - b - c}{3} + \frac{15a - 4c}{12}$.

Ans. $\frac{25a - 20b}{12}$.

15. Simplify $\frac{2x - 3a}{x - 2a} - \frac{2x - a}{x - a}$.

The L. C. D. is $(x - 2a)(x - a)$.

$$\therefore \quad \frac{2x - 3a}{x - 2a} - \frac{2x - a}{x - a}$$

$$= \frac{(2x - 3a)(x - a) - (2x - a)(x - 2a)}{(x - 2a)(x - a)}$$

$$= \frac{2x^2 - 5ax + 3a^2 - (2x^2 - 5ax + 2a^2)}{(x - 2a)(x - a)}$$

$$= \frac{2x^2 - 5ax + 3a^2 - 2x^2 + 5x - 2a^2}{(x - 2a)(x - a)}$$

$$= \frac{a^2}{(x - 2a)(x - a)}$$

16. Simplify $\dfrac{1}{x+y}+\dfrac{x-y}{x^2-xy+y^2}-\dfrac{x^2-xy}{x^3+y^3}$.

The L. C. D. is x^3+y^3.

$$\therefore\ \frac{1}{x+y}+\frac{x-y}{x^2-xy+y^2}-\frac{x^2-xy}{x^3+y^3}$$

$$=\frac{x^2-xy+y^2+(x+y)(x-y)-(x^2-xy)}{x^3+y^3}$$

$$=\frac{x^2-xy+y^2+x^2-y^2-x^2+xy}{x^3+y^3}=\frac{x^2}{x^3+y^3}$$

Simplify:

17. $\dfrac{1}{x+y}+\dfrac{1}{x-y}$.

18. $\dfrac{a}{a-b}-\dfrac{b}{a+b}$.

19. $\dfrac{3}{x-6}-\dfrac{1}{x+2}$.

20. $\dfrac{1+x}{1-x}+\dfrac{1-x}{1+x}$.

21. $\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}$.

22. $\dfrac{a}{x-a}-\dfrac{a^2}{x^2-a^2}$.

23. $\dfrac{1}{2x-3y}-\dfrac{x+y}{4x^2-9y^2}$.

Ans. $\dfrac{x+2y}{4x^2-9y^2}$.

24. $\dfrac{x}{1-x^2}-\dfrac{x}{1+x^2}$.

25. $\dfrac{1}{x(x-y)}+\dfrac{1}{y(x+y)}$.

26. $\dfrac{1}{(x-y)^2}-\dfrac{1}{(x+y)^2}$.

27. $\dfrac{1}{a-2x}-\dfrac{(a+2x)^2}{a^3-8x^3}$.

Ans. $\dfrac{2ax}{8x^3-a^3}$.

28. $\dfrac{y}{x(x^2-y^2)}+\dfrac{x}{y(x^2+y^2)}$.

29. $\dfrac{x^2+xy+y^2}{x+y}+\dfrac{x^2-xy+y^2}{x-y}$.

30. $\dfrac{1}{x^2-7x+12}-\dfrac{1}{x^2-5x+6}$.

31. $\dfrac{x+7}{x^2-3x-10}-\dfrac{x+2}{x^2+2x-35}$.

Ans. $\dfrac{10x+45}{(x+2)(x+7)(x-5)}$.

32. $\dfrac{1}{2x^2-x-1}-\dfrac{3}{6x^2-x-2}$.

33. $\dfrac{1}{2x^2 - x - 1} - \dfrac{1}{2x^2 + x - 3}.$

34. $\dfrac{1}{a + b} + \dfrac{1}{a - b} - \dfrac{2a}{a^2 - b^2}.$

35. $\dfrac{1}{2x + y} + \dfrac{1}{2x - y} - \dfrac{3x}{4x^2 - y^2}.$ Ans. $\dfrac{x}{4x^2 - y^2}.$

36. $\dfrac{a}{a + x} + \dfrac{x}{a - x} - \dfrac{a^2}{a^2 - x^2}.$

37. $\dfrac{5}{1 + 2x} - \dfrac{3x}{1 - 2x} - \dfrac{4 - 13x}{1 - 4x^2}.$

38. $\dfrac{x + y}{x - y} - \dfrac{x - y}{x + y} + \dfrac{4y^2}{x^2 - y^2}.$

39. $\dfrac{a}{a - 1} - 1 - \dfrac{1}{a(a - 1)}.$ Ans. $\dfrac{1}{a}.$

40. $\dfrac{1}{a} - \dfrac{2}{a + 1} + \dfrac{1}{a + 2}.$

41. $\dfrac{1}{x - 1} - \dfrac{1}{2(x + 1)} - \dfrac{x + 3}{2(x^2 + 1)}.$

42. $\dfrac{x}{a + x} + \dfrac{a}{a - x} + 1.$

43. $\dfrac{x}{x - 1} + x - \dfrac{x^2}{x - 1}.$ Ans. $\dfrac{0}{x - 1} = 0.$

44. $\dfrac{3}{x} - \dfrac{5}{2x - 1} - \dfrac{2x - 7}{4x^2 - 1}.$

45. $\dfrac{2}{x + 4} - \dfrac{x - 3}{x^2 - 4x + 16} + \dfrac{x^3}{x^3 + 64}.$

46. $\dfrac{x^3 + ax^2}{ax^2 - a^3} - \dfrac{x(x - a)}{(x + a)a} - \dfrac{2ax}{x^2 - a^2}.$

47. $\dfrac{b}{a + b} - \dfrac{ab}{(a + b)^2} - \dfrac{ab^2}{(a + b)^3}.$ Ans. $\dfrac{b^3}{(a + b)^3}.$

48. $\dfrac{x-y}{x^2-xy+y^2}+\dfrac{1}{x+y}+\dfrac{xy}{x^3+y^3}$.

49. $\dfrac{1}{4a^6(a^2+x^2)}+\dfrac{1}{4a^6(a^2-x^2)}+\dfrac{1}{2a^4(a^4+x^4)}$.

50. $\dfrac{1}{a-x}-\dfrac{1}{a+x}-\dfrac{2x}{a^2+x^2}-\dfrac{4x^3}{a^4+x^4}$. Ans. $\dfrac{8x^7}{a^8-x^8}$.

147. Sometimes a modification of the foregoing general method will considerably shorten the work.

1. Simplify $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1+x^4}$.

$$\frac{1}{1-x}+\frac{1}{1+x}=\frac{2}{1-x^2}$$

then $$\frac{2}{1-x^2}+\frac{2}{1+x^2}=\frac{4}{1-x^4}$$

and finally $$\frac{4}{1-x^4}+\frac{4}{1+x^4}=\frac{8}{1-x^8}$$

2. Simplify $\dfrac{1}{x-3}-\dfrac{3}{x-1}+\dfrac{3}{x+1}-\dfrac{1}{x+3}$.

$$\frac{1}{x-3}-\frac{1}{x+3}=\frac{6}{x^2-9}$$

and $$\frac{3}{x+1}-\frac{3}{x-1}=\frac{-6}{x^2-1}$$

then $$\frac{6}{x^2-9}+\frac{-6}{x^2-1}=\frac{6x^2-6-6x^2+54}{(x^2-9)(x^2-1)}$$

$$=\frac{48}{(x^2-9)(x^2-1)}$$

3. Simplify $\dfrac{a}{x+a}+\dfrac{2x}{x-a}+\dfrac{a(3x-a)}{a^2-x^2}$.

NOTE 1. In working examples like this and like Ex. 4, the symmetric arrangement of the denominators should first be secured by changing the form of any fraction or fractions necessary, in accordance with the principles of § 137 and § 138.

$$\frac{a}{x+a}+\frac{2x}{x-a}+\frac{a(3x-a)}{a^2-x^2}$$

$$=\frac{a}{x+a}+\frac{2x}{x-a}+\frac{a(a-3x)}{x^2-a^2}$$

$$=\frac{a(x-a)+2x(x+a)+a(a-3x)}{x^2-a^2}$$

$$=\frac{ax-a^2+2x^2+2ax+a^2-3ax}{x^2-a^2}$$

$$=\frac{2x^2}{x^2-a^2}$$

4. Simplify $\dfrac{1}{(x-2)(x-3)}-\dfrac{2}{(2-x)(x-1)}+\dfrac{3}{(x-3)(1-x)}$.

$$\frac{1}{(x-2)(x-3)}-\frac{2}{(2-x)(x-1)}+\frac{3}{(x-3)(1-x)}$$

$$=\frac{1}{(x-2)(x-3)}+\frac{2}{(x-1)(x-2)}-\frac{3}{(x-3)(x-1)}$$

$$=\frac{(x-1)+2(x-3)-3(x-2)}{(x-1)(x-2)(x-3)}$$

$$=\frac{x-1+2x-6-3x+6}{(x-1)(x-2)(x-3)}=\frac{-1}{(x-1)(x-2)(x-3)}$$

or $$\frac{1}{(1-x)(x-2)(x-3)}$$

5. Simplify $\dfrac{-1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}$.

NOTE 2. Where the differences of three letters are involved, as in this example, the work is made the shortest and easiest possible by choosing out of the different methods of symmetric arrangement for the denominators what is known as the *cyclic order;* that is, an arrangement where b follows a, c follows b, and a follows c; thus, $a\,b\,c$, $b\,c\,a$, $c\,a\,b$, also $a-b$, $b-c$, $c-a$.

$$\frac{-1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}$$

$$=\frac{1}{(a-b)(c-a)}-\frac{1}{(b-c)(a-b)}-\frac{1}{(c-a)(b-c)}$$

$$=\frac{b-c-(c-a)-(a-b)}{(a-b)(b-c)(c-a)}$$

$$=\frac{b-c-c+a-a+b}{(a-b)(b-c)(c-a)}$$

$$=\frac{2(b-c)}{(a-b)(b-c)(c-a)}=\frac{2}{(a-b)(c-a)}$$

Simplify :

6. $\dfrac{1}{a-x}-\dfrac{1}{a+x}-\dfrac{2x}{a^2+x^2}-\dfrac{4x^3}{a^4+x^4}$. Ans. $\dfrac{8x^7}{a^8-x^8}$.

7. $\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{1}{x-2}-\dfrac{1}{x+2}$.

8. $\dfrac{3}{1+a}-\dfrac{2}{1-a}-\dfrac{5a}{a^2-1}$. Ans. $\dfrac{1}{1-a^2}$.

9. $\dfrac{c}{(a-b)(c+d)}-\dfrac{d}{(b-a)(d+c)}$.

10. $\dfrac{a^2}{(a-b)(c-a)}-\dfrac{b^2-2ab}{(b-a)(b-c)}$.

Ans. $\dfrac{3a^2b-a^2c+b^2c-ab^2-2abc}{(a-b)(b-c)(c-a)}$.

11. $\dfrac{1}{2x+1}+\dfrac{1}{2x-1}+\dfrac{4x}{1-4x^2}$.

12. $\dfrac{1}{x-2}-\dfrac{4}{x-1}+\dfrac{6}{x}-\dfrac{4}{x+1}+\dfrac{1}{x+2}$.

13. $\dfrac{x-2a}{x+a}+\dfrac{2(a^2-4ax)}{a^2-x^2}-\dfrac{3a}{x-a}$. Ans. $\dfrac{x+3a}{x+a}$.

14. $\dfrac{1}{(x-3)(x-4)}-\dfrac{2}{(x-2)(x-4)}+\dfrac{1}{(x-2)(x-3)}$.

15. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}$.

16. $\dfrac{5}{1+2x}-\dfrac{3x}{1-2x}+\dfrac{4-13x}{4x^2-1}$. Ans. $\dfrac{1-6x^2}{1-4x^2}$.

17. $\dfrac{x}{x^3+y^3}-\dfrac{y}{x^3-y^3}+\dfrac{x^3y+xy^3}{x^6-y^6}$.

18. $\dfrac{24x}{9-12x+4x^2}-\dfrac{3+2x}{3-2x}+\dfrac{3-2x}{2x+3}$.

19. $\dfrac{2a}{2a-3}-\dfrac{5}{6a+9}-\dfrac{4(3a+2)}{3(4a^2-9)}$. Ans. $\dfrac{12a^2-4a+7}{3(4a^2-9)}$.

20. $\dfrac{5x}{6(x^2-1)}-\dfrac{1}{2(x-1)}+\dfrac{1}{3(x+1)}$.

21. $\dfrac{3}{8(a-x)}+\dfrac{1}{8(a+x)}-\dfrac{a-x}{4(a^2+x^2)}$.

22. $\dfrac{1}{x+a}+\dfrac{4a}{x^2-a^2}+\dfrac{1}{a-x}-\dfrac{2a}{x^2+a^2}$.

23. $\dfrac{1}{6a-18}-\dfrac{1}{6a+18}-\dfrac{1}{a^2+9}+\dfrac{18}{a^4+81}$.

Ans. $\dfrac{36a^4}{a^8-6561}$.

24. $\dfrac{1}{8-8x}-\dfrac{1}{8+8x}+\dfrac{x}{4+4x^2}-\dfrac{x}{2+2x^4}$.

25. $\dfrac{ax^2+b}{2x-1}+\dfrac{2(bx+ax^2)}{1-4x^2}-\dfrac{ax^2-b}{2x+1}$.

26. $\dfrac{2b-a}{x-b}-\dfrac{3x(a-b)}{b^2-x^2}+\dfrac{b-2a}{x+b}$.

27. $\dfrac{a+c}{(a-b)(x-a)}+\dfrac{b+c}{(b-a)(x-b)}$. Ans. $\dfrac{x+c}{(x-a)(x-b)}$.

28. $\dfrac{a-c}{(a-b)(x-a)} - \dfrac{b-c}{(b-a)(b-x)}$

29. $\dfrac{2a+y}{(x-a)(a-b)} + \dfrac{a+b+y}{(x-b)(b-a)} - \dfrac{x+y-a}{(x-a)(x-b)}$.

30. $\dfrac{1}{(a^2-b^2)(c^2+b^2)} + \dfrac{1}{(b^2-a^2)(c^2+a^2)} - \dfrac{1}{(c^2+a^2)(c^2+b^2)}$.

31. $\dfrac{x-1}{(x-2)(x-3)} + \dfrac{2(x-2)}{(3-x)(x-1)} - \dfrac{x-3}{(x-1)(2-x)}$.

Ans. $\dfrac{2}{(x-1)(x-2)(x-3)}$.

32. $\dfrac{a^2}{(a-b)(a-c)} + \dfrac{b^2}{(b-a)(b-c)} + \dfrac{c^2}{(c-a)(c-b)}$.

33. $\dfrac{a^8}{(a-b)(a-c)} + \dfrac{b^8}{(b-c)(b-a)} + \dfrac{c^8}{(c-a)(c-b)}$.

34. $\dfrac{bc}{(a-b)(a-c)} + \dfrac{ca}{(b-c)(b-a)} + \dfrac{ab}{(c-a)(c-b)}$.

35. $\dfrac{a}{(a-b)(a-c)} + \dfrac{b}{(b-c)(b-a)} - \dfrac{c}{(a-c)(c-b)}$.

Ans. $\dfrac{0}{(a-b)(b-c)(c-a)} = 0$.

36. $\dfrac{1}{a^2(a-b)(a-c)} - \dfrac{1}{b^2(c-b)(b-a)} + \dfrac{1}{c^2(c-a)(c-b)}$.

37. $\dfrac{b}{a(a^2-b^2)} + \dfrac{a}{b(a^2+b^2)} + \dfrac{a^4+b^4}{ab(b^4-a^4)} - \dfrac{a^6}{b^8-a^8}$.

38. $\dfrac{x+1}{x^2+x+1} + \dfrac{x-1}{x^2-x+1} + \dfrac{2}{x^4+x^2+1}$.

39. $\dfrac{1}{x^2-5xy+6y^2} - \dfrac{2}{x^2-4xy+3y^2} + \dfrac{1}{x^2-3xy+2y^2}$.

Ans. $\dfrac{0}{(x-y)(x-2y)(x-3y)} = 0$.

148. MULTIPLICATION AND DIVISION OF FRACTIONS.

Multiplication and division of fractions are simply particular cases of the laws of association and commutation for multiplication and division, as will appear from the following examples.

1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

$$\begin{aligned} \frac{a}{b} \times \frac{c}{d} &= (a \div b) \times (c \div d) \\ &= a \div b \times c \div d \qquad (\S\ 79) \\ &= a \times c \div b \div d \\ &= (a c) \div (b d) \\ &= \frac{a c}{b d} \end{aligned}$$

From this it follows that

$$\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{a c}{b d} \times \frac{e}{f} = \frac{a c e}{b d f}$$

and so for any number of fractions.

Further, $$\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b} = \frac{a a}{b b} = \frac{a^2}{b^2}$$

and in general $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Hence, for the multiplication of fractions, we have the following

Rule.

Multiply the numerators together for a new numerator, and the denominators for a new denominator.

2. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

$$\begin{aligned} \frac{a}{b} \div \frac{c}{d} &= (a \div b) \div (c \div d) \\ &= a \div b \div c \times d \qquad (\S\ 79) \\ &= a \times d \div b \div c \\ &= (a d) \div (b c) \\ &= \frac{a d}{b c} \end{aligned}$$

Hence, for the division of fractions, we have the following

Rule.

Invert the divisor, and then proceed as in multiplication of a fraction by a fraction.

NOTE 1. As a mixed number can be reduced to a fractional form, § 144, and an integral expression can be expressed as a fraction by writing under it 1 as its denominator, this rule covers all possible cases in multiplication and division of fractions.

NOTE 2. *Always cancel, if possible.*

3. Multiply together $\frac{3a^2b}{4c^2d}$, $\frac{6acd^2}{7b^2c^2}$, and $\frac{14cd^3}{9a^2b^2}$.

$$\frac{3a^2b}{4c^2d} \times \frac{6acd^2}{7b^2c^2} \times \frac{14cd^3}{9a^2b^2}$$

$$= \frac{3 \cdot 6 \cdot 14 \cdot a^3bc^2d^5}{4 \cdot 7 \cdot 9 \cdot a^2b^4c^4d} = \frac{ad^4}{b^3c^2}$$

4. Find the value of $\frac{x^2 - y^2}{2xy} \times \frac{x(x+2y)}{xy+y^2} \times \frac{3y^2}{x^2 - xy}$.

$$\frac{x^2 - y^2}{2xy} \times \frac{x(x+2y)}{xy+y^2} \times \frac{3y^2}{x^2 - xy}$$

$$= \frac{(x+y)(x-y)}{2xy} \times \frac{x(x+2y)}{y(x+y)} \times \frac{3y^2}{x(x-y)}$$

$$= \frac{3(x+2y)}{2x}$$

5. Simplify $\frac{6x^2 - ax - 2a^2}{ax - a^2} \times \frac{x-a}{9x^2 - 4a^2} \div \frac{2x+a}{3ax+2a^2}$.

$$\frac{6x^2 - ax - 2a^2}{ax - a^2} \times \frac{x-a}{9x^2 - 4a^2} \div \frac{2x+a}{3ax+2a^2}$$

$$= \frac{6x^2 - ax - 2a^2}{ax - a^2} \times \frac{x-a}{9x^2 - 4a^2} \times \frac{3ax+2a^2}{2x+a}$$

$$= \frac{(3x-2a)(2x+a)}{a(x-a)} \times \frac{x-a}{(3x+2a)(3x-2a)} \times \frac{a(3x+2a)}{2x+a}$$

$$= 1$$

6. Simplify $\left(1+\frac{2x}{a-x}\right)\times\frac{1}{x}\left(a-\frac{x^2}{a}\right)\div\left(1+\frac{a}{x}\right)^2$.

$$\left(1+\frac{2x}{a-x}\right)\times\frac{1}{x}\left(a-\frac{x^2}{a}\right)\div\left(1+\frac{a}{x}\right)^2$$

$$=\frac{a+x}{a-x}\times\frac{a^2-x^2}{a\,x}\div\frac{(x+a)^2}{x^2}$$

$$=\frac{a+x}{a-x}\times\frac{(a+x)(a-x)}{a\,x}\times\frac{x^2}{(x+a)^2}=\frac{x}{a}$$

Simplify the following:

7. $\frac{7a^2c^3}{6bd^4}\times\frac{15b^2cd^2}{28a^5}$.

8. $\frac{12xyz}{25a^3}\times\frac{15a^2x}{16y^2z}\times\frac{10y^3}{9ax^2}$.

9. $\frac{6x^5y}{14a^3b^4}\div\frac{3x^3}{2a^2b^2}$. Ans. $\frac{2x^2y}{7ab^2}$.

10. $\frac{4x^2}{3y^2}\times\frac{12xy}{7z^2}\div\frac{8x^3}{35yz^2}$.

11. $\left(-\frac{2x}{3p}\right)\left(\frac{7p^3}{77q^3}\right)\left(-\frac{3pq^2}{35xy}\right)$.

12. $\frac{x^2-y^2}{x^2}\times\frac{xy}{(x+y)^2}$.

13. $\frac{x^2-4a^2}{ax+2a^2}\times\frac{2a}{x-2a}$. Ans. 2.

14. $\frac{14x^2-7x}{12x^3+24x^2}\div\frac{2x-1}{x^2+2x}$.

15. $\frac{a^2-121}{a^2-4}\div\frac{a+11}{a+2}$.

16. $\frac{25a^2-b^2}{9a^2x^2-4x^2}\times\frac{x(3a+2)}{5a+b}$.

17. $\frac{x^3-a^3}{x^3+a^3}\div\frac{(x-a)^2}{x^2-a^2}$. Ans. $\frac{x^2+xa+a^2}{x^2-xa+a^2}$.

18. $\dfrac{a\,b + b^2}{a^2 - a\,b} \times \dfrac{a^2\,c - b^2\,c}{a^3\,x + b^3\,x}$.

19. $\dfrac{x^2 - 1}{x^2 - 3\,x - 10} \div \dfrac{x^2 - 12\,x + 35}{x^2 + 3\,x + 2}$.

20. $\dfrac{2\,x^2 + 13\,x + 15}{4\,x^2 - 9} \div \dfrac{2\,x^2 + 11\,x + 5}{4\,x^2 - 1}$.

21. $\dfrac{a^4 - b^4}{a^4 + a^2 b^2 + b^4} \div \dfrac{(a - b)^6}{a^6 - b^6}$. Ans. $\dfrac{(a^2 + b^2)\,(a + b)^2}{(a - b)^4}$.

22. $\dfrac{a^3 - 8\,b^3}{a^2 - 4\,b^2} \times \dfrac{a + 2\,b}{a^2 + 2\,a\,b + 4\,b^2}$.

23. $\dfrac{x^3 - 6\,x^2 + 36\,x}{x^2 - 49} \div \dfrac{x^4 + 216\,x}{x^2 - x - 42}$.

24. $\dfrac{a + x}{(a - x)^2} \times \dfrac{a^2 - x^2}{a^2 + x^2} \times \dfrac{a^4 - x^4}{(a + x)^3}$. Ans. 1.

25. $\dfrac{5\,m^6\,n - 5\,n^7}{m^2\,n + 2\,m\,n^2 + n^3} \div \dfrac{m^2 - m\,n + n^2}{m + n}$.

26. $\dfrac{a^2 - a\,b + b^2}{6\,a\,b\,(a^3 + b^3)} \times 4\,a^2\,(a + b)^2$.

27. $\dfrac{x\,a}{x + a} \times \left(\dfrac{x}{a} - \dfrac{a}{x}\right)$. Ans. $x - a$.

28. $\left(a + \dfrac{a\,b}{a - b}\right)\left(b - \dfrac{a\,b}{a + b}\right)$.

29. $\dfrac{x^2 + x\,y}{x^2 + y^2} \times \left(\dfrac{x}{x - y} - \dfrac{y}{x + y}\right)$.

30. $7\,x\,(a - b) \div \dfrac{7\,x}{a^2 - 2\,a\,b + b^2}$.

31. $\dfrac{x^3 - y^3}{2\,y\,(x^2 + y^2)} \times \dfrac{x^4 - 2\,x^2\,y^2 + y^4}{x^4 + x^3\,y + x^2\,y^2} \div \dfrac{x^2 - y^2}{6\,x\,y}$.

Ans. $\dfrac{3\,(x^2 - y^2)\,(x - y)}{x\,(x^2 + y^2)}$.

32. $\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right) \div \left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right).$

33. $\frac{x^2+x-2}{x^2-x-20} \times \frac{x^2+5x+4}{x^2-x} \div \left(\frac{x^2+3x+2}{x^2-2x-15} \times \frac{x+3}{x^2}\right).$

Ans. x.

34. $\left(a+\frac{3x^2}{a}\right) \times \left(\frac{a^2}{3x^2}-1\right) \div \frac{a}{x^2}.$

35. $\frac{12x(x^3+1)}{x^4+x^2} \times \frac{3x^4+6x^2+3}{(x^2-x+1)^2} \div \frac{24(x^2+x)^2}{x^4+1}.$

Ans. $\frac{3(x^2+1)(x^4+1)}{2x^3(x^3+1)}.$

36. $\frac{a^3+b^3+3ab(a+b)}{2a(a-b)^2} \times b^2\left(a-\frac{b^2}{a}\right) \div \left(1+\frac{b}{a}\right)^4.$

37. $\frac{(a+b)^2-c^2}{a^2+ab-ac} \times \frac{a}{(a+c)^2-b^2} \times \frac{(a-b)^2-c^2}{ab-b^2-bc}.$

38. $\frac{a^2+2ab+b^2-c^2}{a^2-b^2-c^2-2bc} \times \frac{c^2-b^2+a^2-2ac}{c^2-a^2+b^2-2bc}.$

39. $\frac{a^4+a^3b+a^2b^2+ab^3}{a^3b+a^2b^2+ab^3+b^4} \times \frac{b^2}{a^3} \div x.$ Ans. $\frac{b}{a^2x}.$

40. $\frac{4a^2+b^2-c^2+4ab}{4a^2-b^2-c^2-2bc} \div \frac{2a+b+c}{2a-b-c}.$

41. $\left(\frac{a}{bc}-\frac{b}{ac}-\frac{c}{ab}-\frac{2}{a}\right) \times \left(1-\frac{2c}{a+b+c}\right).$

42. $\frac{x^4-a^4}{x^6+a^6} \div \left(\frac{x^4-2a^2x^2+a^4}{x^3+a^3} \times \frac{x^3-a^3}{x^4-a^2x^2+a^4}\right).$

Ans. $\frac{x^2-ax+a^2}{(x-a)(x^3-a^3)}.$

149. COMPLEX FRACTIONS.

The division of fractions is sometimes expressed by writing the divisor under the dividend; thus, $\dfrac{\frac{a}{b}}{\frac{x}{y}}$. Such an expression is called a **Complex Fraction.** A complex fraction can be reduced to a simple one by performing the division indicated, or by multiplying its numerator and denominator by the least common multiple of the denominators of the fractional parts.

1. Reduce $\dfrac{\frac{x}{y}}{\frac{9}{5}}$ to a simple fraction.

$$\frac{\frac{x}{y}}{\frac{9}{5}} = \frac{x}{y} \div \frac{9}{5} = \frac{x}{y} \times \frac{5}{9} = \frac{5x}{9y}$$

or,

$$\frac{\frac{x}{y}}{\frac{9}{5}} = \frac{\frac{x}{y} \times 5y}{\frac{9}{5} \times 5y} = \frac{5x}{9y}$$

2. Reduce $\dfrac{\frac{a}{b} + \frac{c}{d}}{\frac{a}{b} - \frac{c}{d}}$ to a simple fraction.

$$\frac{\frac{a}{b} + \frac{c}{d}}{\frac{a}{b} - \frac{c}{d}} = \left(\frac{a}{b} + \frac{c}{d}\right) \div \left(\frac{a}{b} - \frac{c}{d}\right)$$

$$= \frac{ad + bc}{bd} \div \frac{ad - bc}{bd}$$

$$= \frac{ad + bc}{bd} \times \frac{bd}{ad - bc} = \frac{ad + bc}{ad - bc}$$

or,
$$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{a}{b}-\frac{c}{d}}=\frac{\left(\frac{a}{b}+\frac{c}{d}\right)\times b\,d}{\left(\frac{a}{b}-\frac{c}{d}\right)\times b\,d}$$
$$=\frac{a\,d+b\,c}{a\,d-b\,c}$$

3. Reduce $\dfrac{x+\dfrac{a^2}{x}}{x-\dfrac{a^4}{x^3}}$ to its simplest form.

$$\frac{x+\frac{a^2}{x}}{x-\frac{a^4}{x^3}}=\left(x+\frac{a^2}{x}\right)\div\left(x-\frac{a^4}{x^3}\right)$$
$$=\frac{x^2+a^2}{x}\div\frac{x^4-a^4}{x^3}$$
$$=\frac{x^2+a^2}{x}\times\frac{x^3}{x^4-a^4}=\frac{x^2}{x^2-a^2}$$

or
$$\frac{x+\frac{a^2}{x}}{x-\frac{a^4}{x^3}}=\frac{\left(x+\frac{a^2}{x}\right)\times x^3}{\left(x-\frac{a^4}{x^3}\right)\times x^3}=\frac{x^4+a^2x^2}{x^4-a^4}$$
$$=\frac{x^2\,(x^2+a^2)}{(x^2+a^2)\,(x^2-a^2)}=\frac{x^2}{x^2-a^2}$$

4. Simplify $\dfrac{1}{x+1+\dfrac{1}{x-1+\dfrac{1}{x+1+\dfrac{1}{x-1}}}}$.

In simplifying this form of a complex fraction, called a *continued fraction*, we begin at the bottom. The fractions to be simplified *alternate* from a mixed number to a complex fraction, with an integral expression for its numerator and a simple fraction for its denominator. The parts below the oblique lines, which are in many ways helpful to the beginner, show this.

$$\frac{1}{x+1+\dfrac{1}{x-1+\dfrac{1}{x+1+\dfrac{1}{x-1}}}}=\frac{1}{x+1+\dfrac{1}{x-1+\dfrac{1}{\dfrac{x^2}{x-1}}}}$$

$$=\frac{1}{x+1+\dfrac{1}{x-1+\dfrac{x-1}{x^2}}}=\frac{1}{x+1+\dfrac{1}{\dfrac{x^3-x^2+x-1}{x^2}}}$$

$$=\frac{1}{x+1+\dfrac{x^2}{x^3-x^2+x-1}}=\frac{1}{\dfrac{x^4+x^2-1}{x^3-x^2+x-1}}$$

$$=\frac{x^3-x^2+x-1}{x^4+x^2-1},\quad\text{or}\quad\frac{(x-1)\,(x^2+1)}{x^4+x^2-1}$$

Simplify the following:

5. $\dfrac{1+\dfrac{c}{x}}{\dfrac{b}{x}-1}.$

6. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x}-\dfrac{1}{y}}.$

7. $\dfrac{\dfrac{5}{6}x-2}{2}.$

8. $\dfrac{x-\dfrac{3}{7}}{2}-\dfrac{x}{2\frac{1}{3}}.$

9. $\dfrac{\dfrac{4}{x-1}+\dfrac{x-1}{x}}{\dfrac{1}{x-1}-\dfrac{1}{x}}.$

10. $\dfrac{1-\dfrac{2b}{a}+\dfrac{b^2}{a^2}}{1-\dfrac{b^2}{a^2}}.$

11. $\dfrac{x+5+\dfrac{6}{x}}{1+\dfrac{6}{x}+\dfrac{8}{x^2}}.$

Ans. $\dfrac{x\,(x+3)}{x+4}.$

12. $\dfrac{\dfrac{1}{x}-\dfrac{2}{x^2}-\dfrac{3}{x^3}}{\dfrac{9}{x}-x}.$

13. $\dfrac{a+x-\dfrac{a^3}{a^2-ax+x^2}}{a+x-\dfrac{x^3}{a^2-ax+x^2}}.$

14. $$\frac{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}.$$

15. $$\frac{\dfrac{a}{x^2}+\dfrac{x}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ax}+\dfrac{1}{x^2}}.$$

Ans. $a+x$.

16. $$1+\frac{x}{1+x+\dfrac{2x^2}{1-x}}.$$

17. $$\frac{\dfrac{1}{3x-2}-\dfrac{1}{3x+2}}{\dfrac{1}{9-\dfrac{4}{x^2}}}.$$

18. $$\frac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{1-\dfrac{a^2+b^2}{(a+b)^2}}.$$

Ans. $\dfrac{2(a+b)}{a-b}$.

19. $$\frac{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}}{\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}}.$$

20. $$\frac{9x^2-64}{x-1-\dfrac{1}{1-\dfrac{x}{4+x}}}.$$

21. $$\frac{1}{1+\dfrac{1}{1+\dfrac{x}{1-x}}}.$$

22. $$\frac{\dfrac{1}{ab}-\dfrac{1}{ac}-\dfrac{1}{bc}}{\dfrac{a^2-(b-c)^2}{a}}.$$

Ans. $\dfrac{1}{bc(b-a-c)}$.

23. $$\frac{a(a-b)-b(a+b)}{\dfrac{a}{a+b}-\dfrac{b}{a-b}}.$$

24. $$\frac{a^2+b^2}{(a+b)^2}+\frac{\dfrac{2}{ab}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)^3}.$$

25. $$\frac{a+b}{a+b+\dfrac{1}{a-b+\dfrac{1}{a+b}}}.$$

Ans. $\dfrac{a^2-b^2+1}{a^2-b^2+2}$.

26. $$\frac{\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}.$$

27. $$\frac{\dfrac{1+x}{1+x^2}-\dfrac{1+x^2}{1+x^3}}{\dfrac{1+x^2}{1+x^3}-\dfrac{1+x^3}{1+x^4}}.$$

Ans. $\dfrac{1+x^4}{x(1+x^2)}$.

150. MISCELLANEOUS EXAMPLES.

Simplify the following fractions:

1. $\frac{1}{b}\left(\frac{1}{a-b}-\frac{1}{a+2b}\right)-\frac{2}{a^2+ab-2b^2}$.

Ans. $\frac{1}{a^2+ab-2b^2}$.

2. $\frac{a}{b}-\frac{(a^2-b^2)x}{b^2}+\frac{a(a^2-b^2)x^2}{b^2(b+ax)}$.

3. $\frac{4a(a^2-x^2)}{3b(c^2-x^2)}\div\left[\frac{a^2-ax}{bc+bx}\times\frac{a^2+2ax+x^2}{c^2-2cx+x^2}\right]$.

4. $\left\{\frac{x^4-a^4}{x^2-2ax+a^2}\div\frac{x^2+ax}{x-a}\right\}\times\frac{x^5-a^2x^3}{x^3+a^3}\div\left(\frac{x}{a}-\frac{a}{x}\right)$.

5. $\frac{a^2-x^2}{x^2+ax+a^2}\div\frac{\left(1-\frac{x}{a}\right)^3\left(1+\frac{x}{a}\right)^3}{a^3-x^3}$.

Ans. $\frac{a^6}{(a-x)(a+x)^2}$.

6. $\frac{x+3}{2x^2+9x+9}+\frac{1}{2}\cdot\frac{1}{2x-3}-\frac{1}{x-\frac{9}{4x}}$.

7. $\frac{1-a^2}{(1+ax)^2-(a+x)^2}\div\frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)$.

8. $\frac{2x^3+5x^2y+xy^2-3y^3}{3x^4+3x^3y-4x^2y^2-xy^3+y^4}$.

9. $\frac{1}{2}\left(\frac{a^2+x^2}{a^2-x^2}\right)-\frac{1}{2}\cdot\frac{a+x}{a-x}-\left(\frac{a}{a+x}\right)^2$.

Ans. $\frac{a(a^2+x^2)}{(x-a)(x+a)^2}$.

10. $\frac{x}{2}\left(\frac{1}{x-y}-\frac{1}{x+y}\right)\times\frac{x^2-y^2}{x^2y+xy^2}\div\frac{1}{x+y}$.

11. $\frac{1}{x+y}\div\left[\frac{y}{2}\left(\frac{1}{x+y}+\frac{1}{x-y}\right)\times\frac{x^2-y^2}{x^2y+xy^2}\right]$.

12. $\left\{\frac{2}{x}-\frac{1}{a+x}+\frac{1}{a-x}\right\}\div\left(\frac{a+x}{a-x}-\frac{a-x}{a+x}\right)$.

13. $\dfrac{\frac{a-b}{1+ab}+\frac{b-c}{1+bc}}{1-\frac{(a-b)(b-c)}{(1+ab)(1+bc)}}$. Ans. $\frac{a-c}{1+ac}$.

14. $\left\{\dfrac{b+\frac{a-b}{1+ab}}{1-\frac{(a-b)b}{1+ab}}-\dfrac{a-\frac{a-b}{1-ab}}{1-\frac{a(a-b)}{1-ab}}\right\}\div\left(\frac{a}{b}-\frac{b}{a}\right)$.

15. $\dfrac{\frac{x^2+y^2}{y}-x}{\frac{1}{y}-\frac{1}{x}}\times\frac{x^2-y^2}{x^3+y^3}$.

16. $\frac{a^2-b^2-c^2-2bc}{bc+ca+ab}\times\frac{abc}{a^3+b^3+c^3-3abc}$
$\times\frac{a^2+b^2+c^2-ab-bc-ca}{a^2-b^2+c^2-2ca}$.

Ans. $\frac{abc}{(a+b-c)(bc+ca+ab)}$.

17. $\dfrac{1+\frac{x}{1+x}}{x+\frac{1}{1+x}}\div\frac{(x-1)^2-x^2}{x^2+x+1}$.

18. $3 - \cfrac{1}{2 - \cfrac{1}{1 - \cfrac{2x - \frac{1}{3}}{5x}}}$. Ans. $\frac{5}{3x + 2}$.

19. $\cfrac{1}{x - \cfrac{1}{x + \cfrac{1}{x}}} - \cfrac{1}{x + \cfrac{1}{x - \cfrac{1}{x}}}$.

20. $\dfrac{\left(\frac{a + b}{a - b}\right)^2 - \left(\frac{a - b}{a + b}\right)^2}{\left(\frac{a + b}{a - b}\right)^2 + \left(\frac{a - b}{a + b}\right)^2}$. Ans. $\frac{4ab(a^2 + b^2)}{a^4 + 6a^2b^2 + b^4}$.

21. $\dfrac{\frac{1}{2}(2x + 1\frac{1}{3}) - \frac{1}{3}(2 - 3x)}{1\frac{3}{4} - \frac{1}{3}(2x + 4\frac{1}{4})}$. Ans. $\frac{6x}{1 - 2x}$.

22. $\left(\frac{6x}{3x + 4} + \frac{4}{3x - 4} + \frac{16}{16 - 9x^2}\right)$

$\div \left(\frac{1}{4 + 3x} + \frac{3x}{9x^2 - 16}\right)$. Ans. $3x$.

23. $\frac{a^2 - (b - c)^2}{(a + c)^2 - b^2} \times \frac{b^2 - (c - a)^2}{(a + b)^2 - c^2} \times \frac{c^2 - (a - b)^2}{(b + c)^2 - a^2}$.

24. $\left(\frac{1 + x}{1 - x} + \frac{4x}{1 + x^2} - \frac{8x}{x^4 - 1} - \frac{1 - x}{x + 1}\right)$

$\div \left(\frac{1 + x^2}{1 - x^2} + \frac{4x^2}{1 + x^4} - \frac{1 - x^2}{x^2 + 1}\right)$.

Ans. $\frac{2(1 + x^4)}{x}$.

25. $\frac{x^{2n}}{x^n - 1} - \frac{x^{2n}}{x^n + 1} + \frac{1}{1 - x^n} + \frac{1}{x^n + 1}$.

26. $\left(2-\dfrac{3n}{m}+\dfrac{9n^2-2m^2}{m^2+2mn}\right)\div\left(\dfrac{1}{m}-\dfrac{1}{m-2n-\dfrac{4n^2}{m+n}}\right).$

27. $2+\dfrac{1}{2-\dfrac{3}{2+\dfrac{1}{2+\dfrac{3-2x}{x-1}}}}.$ Ans. $\dfrac{5x-1}{2x-1}.$

28. Find the value of $\dfrac{x+y-1}{x-y+1}$, when $x=\dfrac{a+1}{ab+1}$, and $y=\dfrac{a(b+1)}{ab+1}.$ Ans. a.

29. Find the value of $\dfrac{\dfrac{1-a^3}{1-a}+\dfrac{1+a^3}{1+a}}{\dfrac{1-a^3}{1-a}-\dfrac{1+a^3}{1+a}}$, when $a=\dfrac{3}{4}.$

Ans. $\dfrac{25}{12}.$

30. If $a=\frac{1}{3}$, $b=0$, $x=-\frac{1}{3}$, $y=-1$, find the value of $\dfrac{xy-ab}{\dfrac{a^2}{x^2}-\dfrac{a}{x}}-\dfrac{a-x}{a^2+x^2}.$

Simplify:

31. $\dfrac{m-n-2n\dfrac{(m-n)}{m+n}}{\dfrac{m-n}{n}\div\dfrac{m+n}{m}}.$

32. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}.$

Ans. $\dfrac{1}{abc}.$

33. $$\frac{b+c}{a(a-b)(a-c)}+\frac{c+a}{b(b-c)(b-a)}+\frac{a+b}{c(c-a)(c-b)}.$$

34. $$\frac{a-b-c}{(a-b)(a-c)}+\frac{b-c-a}{(b-c)(b-a)}+\frac{c-a-b}{(c-a)(c-b)}.$$

Ans. $\dfrac{0}{(a-b)(b-c)(c-a)}=0.$

35. $$\frac{c+a}{(a-b)(a-c)}+\frac{a+b}{(b-c)(b-a)}+\frac{b+c}{(c-a)(c-b)}.$$

36. $$\frac{3}{(x-y)(y-z)}-\frac{5}{(z-x)(y-x)}-\frac{7}{(x-z)(z-y)}.$$

Ans. $\dfrac{12y-2z-10x}{(x-y)(z-x)(y-z)}.$

37. $$\frac{bc(a+d)}{(a-b)(a-c)}+\frac{ca(b+d)}{(b-c)(b-a)}+\frac{ab(c+d)}{(c-a)(c-b)}.$$

38. $$\frac{\frac{x}{y}+\frac{y}{x}}{\frac{x}{y}-\frac{y}{x}}\div\frac{\frac{x^2}{y^2}-\frac{y^2}{x^2}}{\left(\frac{1}{y}+\frac{1}{x}\right)^2}.$$ Ans. $\dfrac{1}{(x-y)^2}.$

39. $$(a^6-b^6)\,\frac{a-\dfrac{b}{1-\frac{a}{b}}}{a+\dfrac{b}{1+\frac{a}{b}}}.$$

40. $$\frac{xy}{x^2+y^2}\left\{\frac{x+y}{x-y}+\frac{x^3+y^3}{x^3-y^3}\right\}\div\left\{\frac{x+y}{x-y}-\frac{x^3+y^3}{x^3-y^3}\right\}.$$

Ans. 1.

41. $\left(\dfrac{x^2}{y^2}-2+\dfrac{y^2}{x^2}\right)\left(\dfrac{x^4y^4}{xy+y^2}\right)\left(\dfrac{\dfrac{x}{y}-1+\dfrac{y}{x}}{x^3-2x^2y+xy^2}\right).$

42. $\dfrac{\frac{2}{3}(a-1)-\frac{1}{2}\left(2-\frac{a}{3}\right)}{\frac{1}{6}-\frac{1}{2}(1-a)}-\dfrac{\frac{1}{3}\left(1-\frac{a}{2}\right)+\frac{1}{2}\left(\frac{a}{3}-1\right)}{\frac{1}{2}(1-3a)+\frac{1}{3}\left(\frac{3a}{2}+\frac{1}{2}\right)}-1\tfrac{1}{2}.$

Ans. $\dfrac{a-15}{2(3a-2)}.$

43. $\left(\dfrac{a^2}{(x-a)^2}-\dfrac{a^2}{(x+a)^2}+\dfrac{2ax}{x^2-a^2}\right)\div\left(\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}\right).$

Ans. $\dfrac{ax}{x^2-a^2}.$

44. $\dfrac{\dfrac{\frac{1}{x}}{1+\frac{1}{x}}+\dfrac{1-\frac{1}{x}}{\frac{1}{x}}}{\dfrac{\frac{1}{x}}{1+\frac{1}{x}}-\dfrac{1-\frac{1}{x}}{\frac{1}{x}}}.$

45. $\dfrac{\frac{x}{y}+\frac{y}{x}-1}{\frac{x^2}{y^2}+\frac{x}{y}+1}\times\dfrac{1+\frac{y}{x}}{x-y}\div\dfrac{1+\frac{y^3}{x^3}}{\frac{x^2}{y}-\frac{y^2}{x}}.$ Ans. 1.

46. $\dfrac{\frac{1}{x}-\frac{x+a}{x^2+a^2}}{\frac{1}{a}-\frac{a+x}{a^2+x^2}}+\dfrac{\frac{1}{x}-\frac{x-a}{x^2+a^2}}{\frac{1}{a}-\frac{a-x}{a^2+x^2}}.$

47. $\dfrac{\left(\frac{3x+x^3}{1+3x^2}\right)^2-1}{\frac{3x^2-1}{x^3-3x}+1}\div\dfrac{\frac{9}{x^2}-\frac{33-x^2}{3x^2+1}}{\frac{3}{x^2}-\frac{2(x^2+3)}{(x^3-x)^2}}.$ Ans. $\dfrac{x(x+1)}{x^2+4x+1}.$

48. $\frac{5a}{2} - \frac{21b}{2} - 3\left[\frac{1}{6}b - 2\left\{\frac{1}{3}a - b - 3\left(\frac{2}{9}a - \frac{5}{6}b\right)\right\}\right].$

Ans. $\frac{1}{2}a - 2b.$

49. $\frac{a^2 + 24bc - 16b^2 - 9c^2}{16b^2 - 9c^2 - 6ac - a^2} \div \frac{a + 4b - 3c}{a + 4b + 3c}.$

50. $\frac{1}{1 + \frac{1}{1+x}} + \frac{x - (1 - x)}{x + 3 - (1 - 3x)} + \frac{10x^2 - x - 3}{4x^2 + 4x + 1}.$

Ans. $\frac{16x^2 + 23x - 12}{2(x+2)(2x+1)}.$

51. $\frac{8}{b - c} \div \left[\frac{6(a+c)}{7(b^2 - c^2)} \times \left\{\frac{14(b+c)}{9(a-b)} \div \frac{a^2 - c^2}{a^2 - b^2}\right\}\right].$

Ans. $\frac{6(a-c)}{a+b}.$

52. $\frac{4y(x^2 - xy + y^2)}{x^3 + y^3} - \frac{2x^3 - x^2y - 3xy^2}{x^3 + y^3} - \frac{4y}{x+y}$

$+ \frac{2x^2 - 3xy}{x^2 - xy + y^2}.$

53. $\frac{a^3 + b^3}{a^4 - b^4} - \frac{a+b}{a^2 - b^2} - \frac{1}{2}\left(\frac{a-b}{a^2 + b^2} - \frac{1}{a-b}\right).$

54. When $x = 1$, $y = -\frac{1}{2}$, $z = 0$, find the value of

$$\frac{x - [y - z - \{2x - 2y - \frac{1}{2}(3z - y)\}]}{x - \frac{y}{z - \frac{1}{x}}}.$$

CHAPTER XI.

SIMPLE EQUATIONS.

151. AN **Equation** is a statement of the equality of two expressions.

The parts of an equation to the left and right of the sign of equality are called *members*, or *sides*, and are distinguished as the *first member* and *second member*, or the *left side* and *right side*.

152. An **Identical Equation,** or simply an **Identity**, is a statement which is true for *all values* of the letters involved; as, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

153. An **Equation of Condition** is a statement which is only true for *particular* values of the letters involved; as, $x - 3 = 5$ is true only when $x = 8$.

154. A **Literal Equation** is one in which some or all of the known numbers are represented by letters; as,

$$ax - 2b = 5bx - 3a.$$

155. The **Degree** of an equation, when in its simplest form, is shown by the sum of the indices of the unknown factors in that term in which this sum is the greatest.

156. A **Simple Equation** is an equation of the *first degree;* as, $5x - ax = 12$.

157. A **Quadratic Equation** is an equation of the *second degree;* as, $x^2 + 2x = 8$, or $3x^2 - 2xy = 7$.

158. A **Cubic Equation** is an equation of the *third degree;* as, $x^3 + 2x^2y = 5$.

159. The letter whose value it is required to find is called the *unknown number.* The process of finding its value is called *reducing the equation.* The value so found is called the *root* of the equation.

160. If, on the substitution of the root for its unknown symbol, the equation becomes an Identity, the root is said to be *verified.*

161. In the reduction of an equation the processes involved depend upon the Axioms given in § 36. These processes can be best understood by considering an equation as a pair of scales which balance as long as an equal weight remains in both sides; whenever on one side any additional weight is put in or taken out, an equal weight must be put in or taken out on the other side, in order that the equilibrium may remain. So, *in an equation, whatever is done to one side must be done to the other,* in order that the equality may remain. That is,

1. If anything is added to one member, an equal amount must be added to the other.

2. If anything is subtracted from one member, an equal amount must be subtracted from the other.

3. If one member is multiplied by any number, the other member must be multiplied by an equal number.

4. If one member is divided by any number, the other member must be divided by an equal number.

5. If one member is involved or evolved, the other must be involved or evolved to the same degree.

TRANSPOSITION.

162. Transposition is the changing of terms from one member of an equation to the other, without destroying the equality.

This is effected through the use of Axioms 1 and 2, as will appear from the following processes:

$$(1.)\quad \left\{\begin{array}{lr} \text{Let} & x + a = b \\ \text{Now} & a = a \\ \hline \text{By subtraction} & x = b - a \end{array}\right. \quad \text{(Ax. 2.)}$$

$$(2.)\quad \left\{\begin{array}{lr} \text{Let} & x - a = c \\ \text{Now} & a = a \\ \hline \text{By addition} & x = c + a \end{array}\right. \quad \text{(Ax. 1.)}$$

It appears from these examples that any term, as a, which disappears from one member of an equation, reappears in the other with the opposite sign. Hence, *any term may be transposed from one member of an equation to the other, provided its sign is changed.*

It follows from this that *the signs of all the terms of an equation may be changed without destroying the equality;* for this is equivalent to transposing all the terms, and then making the right and left hand members change places. For example:

$$\begin{array}{lrcl} \text{Let} & 5x - 7 &=& 7x - 15 \\ \text{Transposing,} & -7x + 15 &=& -5x + 7 \\ \text{or,} & -5x + 7 &=& -7x + 15 \end{array}$$

The same result could be obtained by either multiplying or dividing the equation throughout by -1. (Ax. 3 or 4.)

REDUCTION OF SIMPLE EQUATIONS CONTAINING BUT ONE UNKNOWN NUMBER.

163. When the Equations are Integral.

1. Reduce $8x + 7 = 4x + 39$.

$$\begin{array}{lrcll} \text{Transposing,} & 8x - 4x &=& 39 - 7 & \\ \text{Combining like terms,} & 4x &=& 32 & \\ \text{Dividing both sides by 4,} & x &=& 8 & \text{(Ax. 4.)} \end{array}$$

Hence, for the reduction of simple equations containing but one unknown number, we have the following

Rule.

Transpose the known terms to one member and the unknown to the other. Combine like terms, and divide both members by the coefficient of the unknown number.

2. Reduce $5(8-x)-3(60-5x)=2(4-x)-40.$

Removing brackets,

$$40-5x-180+15x=8-2x-40$$

Transposing, $$-5x+15x+2x=-40+180+8-40$$

Uniting terms, $$12x=108$$

Dividing by 12, $$x=9$$

3. Reduce $5x-(4x-7)(3x-5)=6-3(4x-9)(x-1).$

Removing brackets,

$$5x-12x^2+41x-35=6-12x^2+39x-27$$

or, $$5x+41x-35=6+39x-27$$

Transposing, $$5x+41x-39x=35+6-27$$

Uniting, $$7x=14$$

$$x=2$$

4. Reduce $7x-5[x-\{7-6(x-3)\}]=3x+1.$

Removing brackets,

$$7x-5x+35-30x+90=3x+1$$

Transposing, $$7x-5x-30x-3x=-35-90+1$$

Uniting, $$-31x=-124$$

$$x=4$$

The student should *verify* his results, especially when the answers are not given, as explained in § 160. For example, if we substitute 8 for x in Example 1, we have

$$8\times 8+7=4\times 8+39$$

that is, $$64+7=32+39$$

or, $$71=71,\ \text{an identity.}$$

Again, if we substitute 2 for x in Example 3, we have

$$10-(8-7)(6-5)=6-3(8-9)(2-1)$$

that is, $$10-1=6-3(-1)$$

or, $$9=9, \text{ an identity.}$$

Reduce the following equations:

5. $12x+7=9x+13.$

6. $17x-50=2x-5.$

7. $12x+4+15x=8+11x+28.$

8. $21-7x=2x+57.$

9. $29x-11=19-11x.$

10. $15-67-18x+12-21=2.$

11. $42x-11x+100+13x-121-23=0.$

12. $8(x-3)-(6-2x)=2(x+2)-5(5-x).$

13. $157-21(x+3)=163-15(2x-5).$

14. $179-18(x-10)=158-3(x-17).$ Ans. 10.

15. $2(x-1)-3(x-2)+4(x-3)+2=0.$

16. $5x+6(x+1)-7(x+2)-8(x+3)=0.$

17. $3(2x-1)-4(6x-5)=12(4x-5)-22.$

18. $x-[3+\{x-(3+x)\}]=5.$ Ans. 5.

19. $25x-19-[3-\{4x-5\}]=3x-(6x-5).$

20. $(x+1)^2-(x^2-1)=x(2x+1)-2(x+2)(x+1)+20.$

21. $10(2x-9)-7(4x-19)+5=4x-3(2x-3).$

22. $20(2-x)+3(x-7)-2[x+9-3\{9-4(2-7)\}]=22.$

23. $3(5-6x)-5[x-5\{1-3(x-5)\}]=23.$ Ans. 4.

24. $3(x-1)^2-3(x^2-1)=x-15.$

25. $2x^2=(x+1)^2+(x+3)^2.$

26. $3x^2 = (x+1)^2 + (x+2)^2 + (x+3)^2.$

27. $(x-2)(x-5) + (x-3)(x-4) = 2(x-4)(x-5).$

28. $(x-1)^2 + 4(x-3)^2 = 5(x+5)^2.$

29. $5(x+1)^2 + 7(x+3)^2 = 12(x+2)^2.$

30. $(x-1)(x-4) = 2x + (x-2)(x-3).$

31. $(x-1)^3 + (x-2)^3 + (x-3)^3 = 3(x-1)(x-2)(x-3).$

32. $4(x-6) - 2\{3x-(x-8)\} = 5(13-3x).$

33. $2(3x+4) = 5\{2x-3(x+4)+9\} - (1-3x).$

Ans. $-3.$

34. $8\{2(x-1)-(x+3)\} = 5[x+7\{x-2(4-x)\}].$

164. Equations with Fractional Terms.

1. Reduce $\frac{x}{5} + \frac{x}{2} - 4 = \frac{3x}{10} - \frac{x}{4} + 2\frac{1}{2}.$

Multiplying every term by 20, the L. C. D.,

$$4x + 10x - 80 = 6x - 5x + 50 \qquad \text{(Ax. 3.)}$$
$$4x + 10x - 6x + 5x = 80 + 50$$
$$13x = 130$$
$$x = 10$$

Hence, for the solution of fractional equations, we have the following

Rule.

Multiply each term of the equation by the least common multiple of the denominators, and then proceed as in the preceding article.

NOTE 1. In multiplying a fractional term, divide the multiplier by the denominator of the fraction and multiply the numerator by the quotient.

NOTE 2. Before clearing of fractions it is better to unite terms which can readily be united.

NOTE 3. When the sign — is before a fraction and the denominator is removed, as the dividing line has the same effect as a bracket, the sign of each term that was in the numerator must be changed.

2. Reduce $\frac{2x}{5} - \frac{x}{3} + 20 = 30 - \frac{x-3}{6}$.

Transposing 20, $$\frac{2x}{5} - \frac{x}{3} = 10 - \frac{x-3}{6}$$

Multiplying by 30, $$12x - 10x = 300 - 5x + 15$$

$$12x - 10x + 5x = 300 + 15$$

$$7x = 315$$

$$x = 45$$

NOTE 4. The sign of the numerator of $-\frac{x}{3}$ is $+$, and must be changed to $-$ when the denominator is removed; for $-(+10x) = -10x$; and so the sign of each term of the numerator of the fraction $-\frac{x-3}{6}$ must be changed when the denominator 6 is removed; for $-(+5x-15) = -5x+15$.

3. Reduce $\frac{x+6}{11} - \frac{2x-18}{3} + \frac{2x+3}{4} = 5\frac{1}{3} + \frac{3x+4}{12}$.

Multiplying by 12, $$\frac{12x+72}{11} - 8x + 72 + 6x + 9 = 64 + 3x + 4$$

Transposing and uniting, $$\frac{12x+72}{11} - 5x = -13$$

Multiplying by 11, $$12x + 72 - 55x = -143$$

$$43x = 215$$

$$x = 5$$

NOTE 5. It will be seen from the above, that it is sometimes advantageous to clear of fractions at first *partially*, then transpose and unite like terms before the remaining fractions are removed.

Reduce the following equations:

4. $\frac{x}{4} + \frac{x-5}{3} = 10.$

5. $\frac{x-5}{10} + \frac{x+5}{5} = 5.$

6. $\frac{x+5}{6} - \frac{x+1}{9} = \frac{x+3}{4}.$

7. $\frac{4-5x}{6} - \frac{1-2x}{3} = \frac{13}{42}.$ Ans. $\frac{1}{7}$.

8. $\dfrac{4(x+2)}{3} - \dfrac{6(x-7)}{7} = 12.$

9. $1 + \dfrac{x}{2} - \dfrac{2x}{3} = \dfrac{3x}{4} - 4\frac{1}{2}.$

10. $\dfrac{6x+7}{5} - \dfrac{2x-1}{10} = 4\frac{1}{2}.$

11. $-\dfrac{7x}{10} + 1\frac{1}{2} + \dfrac{3x}{4} = 2\frac{5}{6} - \dfrac{8x}{15} - 1\frac{1}{3}.$ Ans. 0.

12. $\frac{2}{3}(4x-1) - \frac{1}{7}(3x+2) = 6 + \frac{1}{9}(5x-2).$

13. $\frac{1}{2}(4x-11) + \frac{1}{4}(3x-4) = 3\frac{1}{3} - \frac{1}{12}(3x+13).$

14. $\dfrac{7x}{5} - \dfrac{1}{14}(x-11) = \dfrac{3}{7}(x-25) + 34.$

15. $3 + \dfrac{x}{4} = \dfrac{1}{2}\left(4 - \dfrac{x}{3}\right) - \dfrac{5}{6} + \dfrac{1}{3}\left(11 - \dfrac{x}{2}\right).$ Ans. $3\frac{1}{7}$.

16. $\dfrac{1}{3}\left(\dfrac{x}{4} - 3\right) + \dfrac{5x}{6} - \dfrac{5x}{4} = \dfrac{x-12}{5} - \dfrac{x+3}{3}.$

17. $x - \left(3x - \dfrac{2x-5}{10}\right) = \dfrac{1}{6}(2x-57) - \dfrac{5}{3}.$

18. $\dfrac{x}{4} - \dfrac{x+10}{5} + 4\frac{3}{4} = x - 1 - \dfrac{x-2}{3}.$

19. $\dfrac{5x-8}{2} - \dfrac{10x-7}{3} + 2 = \dfrac{5x}{4} - 1\frac{1}{3}.$

20. $4x - 3\{5x - 8(x + \frac{1}{2})\} = 12x + 10.$ Ans. -2.

21. $12\left(x + \dfrac{1}{3}\right) - (2x+5) = 15x - \dfrac{3(x+3)}{2}.$

22. $3(2x-3) - \dfrac{x+7}{2} + \dfrac{3x-1}{5} = \dfrac{x-4}{4}.$

23. $2(x-1) + \dfrac{3x-5}{4} - \dfrac{4x-3}{9} + \dfrac{x-11}{2} = 0.$

24. $\frac{1}{5}(x-1) + \frac{1}{7}(2x-3) - 3 = 2(x+5).$

25. $\frac{3}{4}(5x+1)-\frac{2}{9}(7x-2)-\frac{2x}{3}=\frac{1}{6}(x-1).$ Ans. -1.

26. $2(3x+1)-\frac{2}{5}(x+4)+20+\frac{2}{3}(x+7)=0.$

27. $5x+\frac{3x-4}{6}-\frac{2}{5}(6x-1)-2(2+x)+5=0.$

28. $4(x+6)-\frac{x}{3}-\frac{2}{5}(x+10)=\frac{13}{10}(x+5)-1\frac{1}{4}.$

29. $5\{2x+1-3(x+1)\}-\frac{x+2}{3}+\frac{10}{7}(3x+5\frac{1}{2})+2\frac{3}{5}=0.$ Ans. $-\frac{1}{5}$.

30. $0.5x+2-0.75x=0.4x-11.$

Writing the decimals as common fractions,

$$\tfrac{1}{2}x+2-\tfrac{3}{4}x=\tfrac{2}{5}x-11$$

Multiplying by 20, the L. C. D.,

$$10x+40-15x=8x-220$$
$$13x=260$$
$$x=20$$

Or,

Transposing, $0.5x-0.75x-0.4x=-2-11$

Combining like terms, $0.65x=13$

Dividing by 0.65, $x=20$

31. $0.5x-0.25x=1.5.$

32. $2.25x-0.125=3x+3.75.$

33. $3+\frac{x}{0.5}=7-\frac{x}{0.2}.$ Ans. $\frac{4}{7}$.

34. $0.6x+6.3-3.5x=0.25x.$

35. $0.6x-0.7x+0.75x-0.875x+15=0.$

36. $0.125x-0.0625x=0.375.$ Ans. 6.

37. $\frac{x+0.75}{0.125}-\frac{x-0.25}{0.25}=15.$

38. $0.5\,x - \dfrac{0.45\,x - 0.75}{0.6} = \dfrac{1.2}{0.2} - \dfrac{0.3\,x - 0.6}{0.9}$.

39. $1.5 = \dfrac{0.36}{0.2} - \dfrac{0.09\,x - 0.18}{0.9}$. Ans. 5.

165. When the Equations are Literal.

1. Reduce $2\,x\,(a + b) - 3\,a\,b = 2\,a\,(x - b) - b^2$.

Removing the brackets,

$$2\,a\,x + 2\,b\,x - 3\,a\,b = 2\,a\,x - 2\,a\,b - b^2$$

Dropping $2\,a\,x$ from both members, transposing, and uniting,

$$2\,b\,x = a\,b - b^2$$

$$x = \frac{b\,(a - b)}{2\,b} = \frac{a - b}{2}$$

2. Reduce $\dfrac{a\,b + x}{b^2} - \dfrac{b^2 - x}{a^2\,b} = \dfrac{x - b}{a^2} - \dfrac{a\,b - x}{b^2}$.

Multiplying by $a^2 b^2$, the L. C. D.,

$$a^3\,b + a^2\,x - b^3 + b\,x = b^2\,x - b^3 - a^3\,b + a^2\,x$$

Dropping $a^2\,x$ and $-\,b^3$ from both members, and transposing,

$$b\,x - b^2\,x = -\,a^3\,b - a^3\,b$$

$$(b - b^2)\,x = -\,2\,a^3\,b$$

$$x = \frac{-\,2\,a^3\,b}{b - b^2} = \frac{-\,2\,a^3}{1 - b} = \frac{2\,a^3}{b - 1}$$

Reduce the following equations:

3. $2\,(x - a) + 3\,(x - 2\,a) = 2\,a$.

4. $\frac{1}{2}\,(x + a + b) + \frac{1}{3}\,(x + a - b) = b$.

5. $(a + b)\,x + (a - b)\,x = a^2$. Ans. $\dfrac{a}{2}$.

6. $(a + b)\,x + (b - a)\,x = b$.

7. $\frac{1}{2}\,(a + x) + \frac{1}{3}\,(2\,a + x) + \frac{1}{4}\,(3\,a + x) = 3\,a$.

8. $\dfrac{x\,a}{b} + \dfrac{x\,b}{a} = a^2 + b^2$. Ans. $a\,b$.

9. $(a + bx)(b + ax) = ab(x^2 - 1)$.

10. $a(x + a) + b(b - x) = 2ab$.

11. $(x + a + b)^2 + (x + a - b)^2 = 2x^2$.

12. $(x - a)(x - b) + (a + b)^2 = (x + a)(x + b)$.
Ans. $\frac{1}{2}(a + b)$.

13. $(x+a+b+c)(x+a-b-c) = (x-a-b+c)(x-a+b-c)$.

14. $ax(x + a) + bx(x + b) = (a + b)(x + a)(x + b)$.

15. $(x - a)^3 + (x - b)^3 + (x - c)^3 = 3(x - a)(x - b)(x - c)$.
Ans. $\frac{1}{3}(a + b + c)$.

16. $\frac{9a}{b} - \frac{3x}{b} = \frac{4b}{a} - \frac{2x}{a}$.

17. $\frac{2}{3}\left(\frac{x}{a} + 1\right) = \frac{3}{4}\left(\frac{x}{a} - 1\right)$.

18. $\frac{1}{4}x(x - a) - \left(\frac{x + a}{2}\right)^2 = \frac{2a}{3}\left(x - \frac{a}{2}\right)$.

19. $b(a - x) - \frac{a}{b}(b + x)^2 + ab\left(\frac{x}{b} + 1\right)^2 = 0$. Ans. a.

20. $x^2 + a(2a - x) - \frac{3b^2}{4} = \left(x - \frac{b}{2}\right)^2 + a^2$. Ans. $a + b$.

166. Since, in the solution of problems, the relations of the numbers involved are often best expressed in the form of a proportion, we introduce here the necessary definitions.

167. Ratio is the relation of one number to another of the same kind; or, it is the quotient which arises from dividing one number by another of the same kind.

Ratio is indicated by writing the two numbers one after the other with two dots between, or by expressing the

division in the form of a fraction. Thus, the ratio of a to b is written, $a:b$, or $\frac{a}{b}$; read, a is to b, or a divided by b.

168. The **Terms** of a ratio are the quantities compared, whether simple or compound.

The first term of a ratio is called the *antecedent*, and the other the *consequent;* and the two terms together are called a *couplet.*

169. Proportion is an equality of ratios. Four numbers are proportional when the ratio of the first to the second is equal to the ratio of the third to the fourth.

The equality of two ratios is indicated by the sign of equality (=) or by four dots (: :). Thus, $a:b=c:d$, or $a:b::c:d$, or $\frac{a}{b}=\frac{c}{d}$; read, a to b equals c to d, or a is to b as c is to d, or a divided by b equals c divided by d.

In a proportion the antecedents and consequents of the two ratios are respectively the *antecedents* and *consequents* of the proportion. The first and fourth terms are called the *extremes*, and the second and third the *means.*

170. *If four numbers are in proportion, the product of the means is equal to the product of the extremes.*

Let $$a:b=c:d$$

that is, $$\frac{a}{b}=\frac{c}{d}$$

Clearing of fractions, $$ad=bc$$

Hence, if any three terms of a proportion are given, the fourth may be found. Thus, if a, c, d are given, then $b=\frac{ad}{c}$.

A proportion is an equation; and making the product of the means equal to the product of the extremes is merely clearing the equation of fractions.

171. QUESTIONS LEADING UP TO PROBLEMS.

1. Form a proportion out of the following numbers: 3, 9, 15, 5.

2. Arrange the ratios $5:6$, $7:8$, $41:48$, $31:36$, in the order of their magnitude.

3. For what value of x will the ratio $3+x:4+x$ be equal to $8:9$?

4. What number must be added to each term of the ratio $11:17$ to make it equal to the ratio of $2:3$?

5. Write four consecutive numbers, of which x is the least.

6. Write three consecutive numbers, of which y is the greatest.

7. Write five consecutive numbers, of which x is the middle one.

8. What is the next even number after $2n$?

9. What is the odd number next before $2x+1$?

10. Find the sum of three consecutive odd numbers, of which the middle one is $2n+1$.

11. A horse eats a bushels and a donkey b bushels of corn in a week. How many bushels will they together consume in n weeks?

12. If a man was x years old five years ago, how old will he be y years hence?

13. A boy is x years old, and five years hence his age will be half that of his father. How old is the father now?

14. What is the age of a man who y years ago was m times as old as a child then x years old?

15. A's age is double B's, B's is three times C's, and C is x years old. Find A's age.

16. Find a number exceeding the sum of x and y by their difference.

17. A post $6x$ inches in length is half in the mud, and one sixth in the water. How many inches are above the water?

18. There are two casks containing 20 gallons each. If x gallons are poured from the first to the second, how many gallons will each cask contain?

19. I pour x gallons of milk and y of water into a bucket. What fraction of the whole is milk, and what water?

20. If I rowed x miles in 10 hours, at what rate did I row?

21. A and B, starting from the same point, walk in opposite directions at the rate of x and y miles an hour respectively. How far apart will they be in 20 minutes?

22. A and B, starting from the same point, walk in the same direction at the rate of x and y miles an hour respectively. How far apart will they be at the end of one hour? In three hours?

23. If A gives B a start of one mile, and then overtakes him in x hours, how many miles an hour does A walk faster than B?

24. How many miles can a person walk in 45 minutes, if he walks a miles in x hours?

25. How long will it take a person to walk b miles, if he walks 20 miles in c hours?

26. If a train goes a miles in b hours, how many feet does it move through in one second?

27. A train is running with a velocity of x feet per second. How many miles will it run in y hours?

28. How long will x men take to mow y acres of corn, if each man mows z acres a day?

29. A beats B by x yards in a mile race. At the same rate of running, how far ahead will A be when B comes to the end of the mile?

30. A room is $x + y$ feet long, and $x - y$ feet broad. What is its area?

31. What is the area of a square court, each side of which is $2x$ yards long?

32. A block of buildings $3x$ yards square is placed in the centre of a court $4x$ yards square. Find the area of the unoccupied space.

33. If a room is a feet long, b feet broad, and c feet high, how much carpet would be required for the floor?

34. If a room is p feet long and x yards in width, how many yards of carpet two feet wide will be required for the floor?

35. What is the cost in dollars of carpeting a room a yards long, and b feet broad, with carpet costing c cents a square yard?

36. If A can do a piece of work in x days, what fraction of the work can he do in one day?

37. If A can do a piece of work in $\frac{x}{a}$ days, what fraction of the work can he do in one day?

38. A can do a piece of work in x days, B in y days. What fraction of the work can they do together in one day?

39. If a pipe discharges x gallons of water in y hours, how many does it discharge in one hour?

40. A pipe is letting off water from a full cistern containing z gallons. If it could empty the whole in y hours, how much water would there be in the cistern at the end of one hour? How much at the end of a hours?

41. If a pipe can fill a cistern in y hours, what fraction of the cistern can it fill in three hours?

42. There are x men in each outer face of a hollow square, and y men in each inner face. How many men are there in the square?

43. The digit in the units' place of a number is n, and in the tens' place m. What is the number?

44. The digits of a number beginning from the left are a, b, c. What is the number? If the order of the digits of the number is reversed, what is the number thus formed?

45. A number consists of 4 digits, the first of which is a, the second 0, the third b, and the fourth c. What is the number?

46. Find the interest on $1 for n years at r per cent.

47. Find the amount of $1 for n years at r per cent.

48. If there are 4 stones in a row, what number from the end is the second from the beginning?

49. If there are 20 stones in a row, what numbers from the beginning are the seventh and nth from the end, respectively?

PROBLEMS

PRODUCING EQUATIONS OF THE FIRST DEGREE CONTAINING BUT ONE UNKNOWN NUMBER.

172. The problems given in this chapter must either contain but one unknown number, or the unknown numbers must be so related to one another that if one becomes known the others also become known.

173. With beginners the chief difficulty in solving a problem is in translating the statements or conditions of the problem from common to algebraic language; that is, in preparing the data and forming an equation in accordance with the given conditions.

It is impossible to give a general rule for the solution of problems, applicable to all cases. Each problem must be considered, and its meaning thoroughly understood, before an equation can be formed.

The following suggestions may be of service:

1. *Let x (or some one of the latter letters of the alphabet) represent the unknown number; or, if there is more than*

one unknown number, let x represent one, and find the others by expressing in algebraic form their given relations to the one represented by x.

2. *With the data thus prepared, form an equation in accordance with the conditions given in the problem.*

3. *Reduce the equation.*

The three steps may be briefly expressed thus:

1st. Preparing the data;
2d. Forming the equation;
3d. Reducing the equation.

1. Find two numbers whose sum is 28, and whose difference is 4.

Let $x =$ the smaller number;
then $x + 4 =$ the greater number.

By addition, $x + x + 4 =$ their sum;
but $28 =$ their sum;

$\therefore\ x + x + 4 = 28$

$2x = 24$

$x = 12$, the smaller number,
and $x + 4 = 16$, the greater number.
Verification, $16 + 12 = 28$

2. Divide \$59 among A, B, and C, so that A may have \$9 more than B, and B \$7 more than C.

Let $x =$ the number of dollars in C's part;
then $x + 7 =$ " " " B's "
and $x + 7 + 9 =$ " " " A's "

By addition, $3x + 23 =$ the number of dollars to be divided;
but $59 =$ " " " " "

$\therefore\ 3x + 23 = 59$

$3x = 36$

$x = 12$

C has \$12, B \$19, A \$28.

Verification, $12 + 19 + 28 = 59$

3. Charles is now twice as old as Henry, and eight years ago he was six times as old. What are their present ages?

Let $x =$ the number of years in Henry's age now ;
then $2x =$ " " " Charles's "
$x - 8 =$ the number of years in Henry's age eight years ago,
$2x - 8 =$ " " " Charles's " " "
$6(x - 8) =$ " " " " " " "

$$\therefore\quad 2x - 8 = 6(x - 8)$$
$$2x - 8 = 6x - 48$$
$$-4x = -40$$
$$x = 10$$

Therefore Henry and Charles are 10 and 20 years old, respectively.

4. One number exceeds another by 5, and the sum of the two is 29. What are the numbers?

5. Find three consecutive numbers whose sum is 96.

6. The difference between two numbers is 8, and if 2 be added to the greater the result will be three times the smaller. Find the numbers. Ans. 13, 5.

7. A man walks 10 miles, then travels a certain distance by train, and then twice as far by coach. If the whole journey is 70 miles, how far does he travel by train?

8. If 288 be added to a certain number, the result will be equal to three times the excess of the number over 12. Find the number.

9. Find a number such that, if 5, 15, and 35 are added to it, the product of the first and third results may be equal to the square of the second. Ans. 5.

10. If the difference between the squares of two consecutive numbers is 121, what are the numbers?

11. A father is four times as old as his son, and in 24 years he will be twice as old. Find their ages.

12. A is 25 years older than B, and A's age is as much above 20 as B's is below 85. Find their ages.

Ans. A, 65; B, 40.

13. The length of a room exceeds the breadth 3 feet. If the length is increased 3 feet, and the breadth diminished 2 feet, the area will remain the same. Find the dimensions.

14. The length of a room exceeds the breadth 8 feet. If each dimension is increased 2 feet, the area will be increased 60 square feet. Find the dimensions of the room.

15. Find two numbers which differ by 4, and such that one half the greater exceeds one sixth of the less by 8.

Let x = the smaller number;

then $x + 4$ = the greater number,

$\frac{x}{6}$ = one sixth of the less,

$\frac{x}{2} + 2$ = one half of the greater,

$\frac{x}{2} + 2 - \frac{x}{6}$ = the excess of $\frac{1}{2}$ the greater over $\frac{1}{6}$ the less.

But $8 =$ " " " " " "

$$\therefore \frac{x}{2} + 2 - \frac{x}{6} = 8$$

$$\frac{x}{3} = 6$$

$x = 18$, the less number,

$x + 4 = 22$, the greater.

16. A has \$180, and B \$84. How much must A give to B in order that A may have five sixths as much as B?

Let x = the number of dollars A must give to B;

then $84 + x$ = " " " B will then have,

and $180 - x$ = " " " A " "

But $\frac{5}{6}(84 + x)$ = " " " A " "

$$\therefore 180 - x = \frac{5}{6}(84 + x) = 70 + \frac{5}{6}x$$

$$-\frac{11}{6}x = -110$$

$$x = 60$$

That is, A must give \$60 to B.

17. Find the number whose fifth, fifteenth, and twenty-fifth together are 23. Ans. 75.

18. Four times the difference between the fourth and fifth parts of a certain number exceeds by 4 the difference between the third and seventh parts. What is the number?

19. Find three consecutive numbers such that, if they are divided by 10, 17, and 26 respectively, the sum of the quotients will be 10.

20. From a certain number 3 is taken, and the remainder is divided by 4; the quotient is then increased by 4 and divided by 5 and the result is 2. Find the number. Ans. 27.

21. A sum of money is divided among three persons, A, B, and C, in such a way that A and B have together \$60, A and C \$65, and B and C \$75. How much has each?

22. Four persons, A, B, C, D, have certain sums of money, such that A and B together have \$49, A and C \$51, B and C \$53, and A and D \$47. How much has each?

23. Divide 15 dollars among 3 men, 5 women, and 20 children, giving to each man one dollar more than to each woman, and to each child half as much as to each woman.

24. A man leaves one half of his property to his wife, one third to his son, and the remainder (which is \$2000) to his daughter. How much did he leave in all? Ans. \$12000.

25. A man left his property to be divided among his three children in such a way that the share of the eldest was to be twice that of the second, and the share of the second twice that of the youngest. It was found that the eldest received \$750 more than the youngest. How much did each receive?

26. An estate of 8000 acres is to be divided among three persons, A, B, and C, so that B has 276 acres less than A, and C 1112 acres more than B. How many acres does each get?

Ans. A, 2480; B, 2204; C, 3316.

27. A person has a flock of sheep and goats, together numbering 75. He has two goats to every three sheep. How many are there of each?

28. If \$1200 is divided between A and B in the proportion of 2 to 7, how much does A get less than B?

29. Find a number which, increased by its half, its third, and its fourth, will amount to 50.

30. The width of a room is two thirds of its length. If the width had been 3 feet more, and the length 3 feet less, the room would have been square. Find its dimensions.

31. A spends a sum of money; B spends half as much as A; C spends three fourths as much as A and B together. If A, B, and C together spend \$1050, what does each spend?

32. 2850 acres of land are divided among three persons, A, B, and C, so that the shares of A and B are in the ratio of 6 to 11, and C has 300 acres more than A and B together. What does each get?

33. \$7500 is divided among a mother, two sons, and three daughters, so that each son has twice as much as each daughter, and the mother \$500 more than all the children together. How much does each get? Ans. \$500, \$1000, \$4000.

34. A post is a fourth of its length in the mud, a third of its length in the water, and 10 feet above the water. What is its length?

35. A marketwoman, being asked how many eggs she had, replied, "If I had as many more, and one less than half as many more, I should have 104 eggs." How many had she?

36. Find two numbers, whose difference is 25, such that the second divided by the first gives 4 as quotient and 4 as remainder. Ans. 7, 32.

37. Find a fraction whose denominator exceeds its numerator by 4, while, if the numerator is diminished 1 and the denominator increased 1, the fraction becomes equal to $\frac{1}{2}$.

38. A person paid $800 for 4 horses. For the second he gave half as much again as for the first; for the third, half as much again as for the second; and for the fourth, as much as for the first and third together. What was the cost of the fourth horse? Ans. $325.

174. The following fractional equations are more difficult to reduce than those thus far given, since their denominators are generally polynomials rather than monomials, and generally involve the unknown number.

The method already given still applies, but it is often unnecessarily prolix, and hence for the ready solution of such examples special artifices must be frequently resorted to.

1. Reduce $\dfrac{4}{3x-6}+\dfrac{1}{6-2x}=\dfrac{x+3}{3\,(42-35\,x+7\,x^2)}$.

Factoring the denominators and writing them in symmetrical form,

$$\frac{4}{3\,(x-2)}-\frac{1}{2\,(x-3)}=\frac{x+3}{21\,(x-2)\,(x-3)}$$

Multiplying by $42\,(x-2)\,(x-3)$, the L. C. D.,

$$56\,(x-3)-21\,(x-2)=2\,(x+3)$$

Freeing from brackets,

$$56\,x-168-21\,x+42=2\,x+6$$

$$33\,x=132$$

$$x=4$$

Verification,

$$\frac{4}{3\,(4)-6}+\frac{1}{6-2\,(4)}=\frac{(4)+3}{3\,\{42-35\,(4)+7\,(4)^2\}}$$

That is, $$\frac{1}{6}=\frac{1}{6}$$

2. Reduce $\dfrac{8x+23}{20} - \dfrac{5x+2}{3x+4} = \dfrac{2x+3}{5} - 1.$

Multiplying by 20,

$$8x + 23 - \frac{20(5x+2)}{3x+4} = 8x + 12 - 20$$

Transposing and uniting,

$$\frac{20(5x+2)}{3x+4} = 31$$

Multiplying by $3x+4$, $\quad 100x + 40 = 93x + 124$

$$7x = 84$$

$$x = 12$$

Verification,

$$\frac{8(12)+23}{20} - \frac{5(12)+2}{3(12)+4} = \frac{2(12)+3}{5} - 1$$

That is, $$\frac{22}{5} = \frac{22}{5}$$

3. Reduce $\dfrac{x}{x-2} + \dfrac{x-9}{x-7} = \dfrac{x+1}{x-1} + \dfrac{x-8}{x-6}.$

Transposing, $$\frac{x}{x-2} - \frac{x+1}{x-1} = \frac{x-8}{x-6} - \frac{x-9}{x-7}$$

Combining,

$$\frac{x(x-1)-(x-2)(x+1)}{(x-2)(x-1)} = \frac{(x-7)(x-8)-(x-6)(x-9)}{(x-6)(x-7)}$$

Or, $$\frac{2}{(x-2)(x-1)} = \frac{2}{(x-6)(x-7)}$$

Dividing by 2 and clearing of fractions,

$$(x-2)(x-1) = (x-6)(x-7)$$

$$x^2 - 3x + 2 = x^2 - 13x + 42$$

$$10x = 40$$

$$x = 4$$

Verification, $$\frac{4}{4-2} + \frac{4-9}{4-7} = \frac{4+1}{4-1} + \frac{4-8}{4-6}$$

That is, $$2 + \tfrac{5}{3} = 2 + \tfrac{5}{3}$$

4. Reduce $\frac{x-3}{x+2}=\frac{x-5}{x+6}$.

Reducing the fractions to mixed numbers,

$$1-\frac{5}{x+2}=1-\frac{11}{x+6}$$

$$\therefore \quad \frac{5}{x+2}=\frac{11}{x+6}$$

Clearing of fractions,

$$5x+30=11x+22.$$

$$6x=8$$

$$x=\tfrac{4}{3}$$

5. Reduce $\frac{x^3+7x^2+24x+30}{x^2+5x+13}=\frac{2x^3+11x^2+36x+45}{2x^2+7x+20}$.

Reducing the fractions to mixed numbers,

$$x+2+\frac{x+4}{x^2+5x+13}=x+2+\frac{2x+5}{2x^2+7x+20}$$

Removing $x+2$ from both members, and inverting the fractions,

$$\frac{x^2+5x+13}{x+4}=\frac{2x^2+7x+20}{2x+5}$$

Reducing to mixed numbers,

$$x+1+\frac{9}{x+4}=x+1+\frac{15}{2x+5}$$

$$\therefore \quad \frac{9}{x+4}=\frac{15}{2x+5}$$

$$18x+45=15x+60$$

$$3x=15$$

$$x=5$$

Ex. 3 may be written in the following form, and then solved like Exs. 4 and 5 :

$$\frac{x-2+2}{x-2}+\frac{x-7-2}{x-7}=\frac{x-1+2}{x-1}+\frac{x-6-2}{x-6} \qquad (1)$$

Reducing to mixed numbers,

$$1+\frac{2}{x-2}+1-\frac{2}{x-7}=1+\frac{2}{x-1}+1-\frac{2}{x-6} \qquad (2)$$

That is,

$$\frac{2}{x-2}-\frac{2}{x-7}=\frac{2}{x-1}-\frac{2}{x-6}$$

$$\therefore \quad \frac{-10}{(x-2)(x-7)}=\frac{-10}{(x-1)(x-6)}$$

$$x=4$$

NOTE. Equation (2) can also be found directly, without writing (1).

Reduce the following equations:

6. $\dfrac{2}{4x^2-1}-\dfrac{3}{1-2x}=\dfrac{11}{2x+1}.$

7. $\dfrac{2x+1}{2x-1}+\dfrac{8}{1-4x^2}=\dfrac{2x-1}{2x+1}.$

8. $\dfrac{2}{x-4}-\dfrac{1}{2x-9}=\dfrac{7}{2x^2-17x+36}.$

9. $\dfrac{1}{2(3x+7)}-\dfrac{2}{3x^2+22x+35}+\dfrac{1}{2x+10}=0.$

Ans. -2.

10. $\dfrac{2x-5}{5}+\dfrac{x-3}{2x-15}=\dfrac{4x-3}{10}-1\tfrac{1}{10}.$

11. $\dfrac{4(x+3)}{9}=\dfrac{8x+37}{18}-\dfrac{7x-29}{5x-12}.$

12. $\dfrac{(2x-1)(3x+8)}{6x(x+4)}-1=0.$

13. $\dfrac{4}{x+3}-\dfrac{2}{x+1}=\dfrac{5}{2x+6}-\dfrac{2\frac{1}{2}}{2x+2}.$

14. $\dfrac{7}{x-4} - \dfrac{60}{5x-30} = \dfrac{10\frac{1}{2}}{3x-12} - \dfrac{8}{x-6}$. Ans. -10.

15. $\dfrac{25 - \frac{x}{3}}{x+1} + \dfrac{16x + 4\frac{1}{5}}{3x+2} = 5 + \dfrac{23}{x+1}$.

16. $\dfrac{30 + 6x}{x+1} + \dfrac{60 + 8x}{x+3} = 14 + \dfrac{48}{x+1}$.

17. $\dfrac{1}{x-1} - \dfrac{1}{x-2} = \dfrac{1}{x-3} - \dfrac{1}{x-4}$.

18. $\dfrac{x-7}{x-9} - \dfrac{x-9}{x-11} = \dfrac{x-13}{x-15} - \dfrac{x-15}{x-17}$.

19. $\dfrac{2x+1}{x+1} + \dfrac{2x+9}{x+5} = \dfrac{2x+3}{x+2} + \dfrac{2x+7}{x+4}$. Ans. -3.

20. $\dfrac{4x-17}{x-4} + \dfrac{10x-13}{2x-3} = \dfrac{8x-30}{2x-7} + \dfrac{5x-4}{x-1}$.

21. $\dfrac{5x-8}{x-2} + \dfrac{6x-44}{x-7} - \dfrac{10x-8}{x-1} = \dfrac{x-8}{x-6}$.

22. $\dfrac{x^2+2x-2}{x-1} + \dfrac{x^2-2x-2}{x+1} = \dfrac{2x^2-6x+2}{x-3}$.

23. $\dfrac{4x^3+4x^2+8x+1}{2x^2+2x+3} = \dfrac{2x^2+2x+1}{x+1}$. Ans. 2.

24. $\dfrac{x+4a+b}{x+a+b} + \dfrac{4x+a+2b}{x+a-b} = 5$.

25. $\dfrac{2x+3a}{x+a} = \dfrac{2(3x+2a)}{3x+a}$.

26. $\dfrac{a}{x-a} - \dfrac{b}{x-b} = \dfrac{b^2-a^2}{b^2-bx}$.

27. $(2x-a)\left(x+\frac{2a}{3}\right)=4x\left(\frac{a}{3}-x\right)-\frac{1}{2}(a-4x)(2a+3x).$

Ans. $\frac{2a}{21}$.

28. $\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)b^2x}{a(a+b)^2}=3cx+\frac{bx}{a}.$

29. $\frac{2x-3}{0.3x-0.4}=\frac{0.4x-6}{0.06x-0.07}.$

30. $\frac{1-1.4x}{0.2+x}=\frac{0.7(x-1)}{0.1-0.5x}.$

31. $\frac{(0.3x-2)(0.3x-1)}{0.2x-1}-\frac{1}{6}(0.3x-2)=0.4x-2.$

Ans. 20.

32 $\frac{x+10}{3}-\frac{2}{5}(3x-4)+\frac{(3x-2)(2x-3)}{6}=x^2-\frac{2}{15}.$

33. $\frac{x+m}{x^2+mx+m^2}-\frac{x-m}{x^2-mx+m^2}=\frac{m^4}{x(x^4+m^2x^2+m^4)}.$

34. $\frac{x-a}{x-a-1}-\frac{x-a-1}{x-a-2}=\frac{x-b}{x-b-1}-\frac{x-b-1}{x-b-2}.$

Ans. $\frac{1}{2}(a+b+3)$.

35. $\frac{x-\frac{1}{2}}{x-\frac{3}{2}}+\frac{x-\frac{13}{2}}{x-\frac{15}{2}}=\frac{x-\frac{3}{2}}{x-\frac{5}{2}}+\frac{x-\frac{11}{2}}{x-\frac{13}{2}}.$

36. $\frac{\frac{1}{8}(x-2)}{\frac{2}{3}}+\frac{\frac{3}{4}(6x-7)}{5}=\frac{\frac{2x}{3}-1}{\frac{1}{6}}-1\frac{1}{4}.$

37. $\frac{1}{2}x-\frac{\frac{1}{3}(2x-3)-\frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)}=\frac{3}{2}\cdot\frac{x^2-\frac{1}{3}x+2}{3x-2}.$

Ans. 0.

175. ADDITIONAL PROBLEMS.

1. A man's age was to that of his wife at the time of their marriage as $3:2$, and their ages 9 years after were as $4:3$. What was the age of each at the time of their marriage?

Let $3x =$ No. of yrs. in the man's age at marriage;
then $2x =$ " " wife's " " .
$3x + 9 =$ No. of yrs. in the man's age 9 yrs. after marriage,
$2x + 9 =$ " " wife's " " "

By the conditions of the question,

$$3x + 9 : 2x + 9 = 4 : 3$$

$$\therefore \quad 9x + 27 = 8x + 36$$

$$x = 9$$

$$3x = 27, \text{ man's age,}$$

$$2x = 18, \text{ wife's age.}$$

Verify the example.

2. A can do a piece of work in 3 days, and B can do it in 5 days. In what time can they do it together?

Let $x =$ the number of days it will take A and B together.

then $\frac{1}{x} =$ the part A and B can do in one day.

Now $\frac{1}{3} =$ the part A can do in one day,

and $\frac{1}{5} =$ " B " "

$\therefore \frac{1}{3} + \frac{1}{5} =$ the part A and B can do in one day,

or, $$\frac{1}{3} + \frac{1}{5} = \frac{1}{x}$$

$$\frac{8}{15} = \frac{1}{x}$$

$$x = \frac{15}{8} = 1\frac{7}{8}$$

3. Two trains are running on parallel tracks, in the same direction, one at the rate of 30 miles an hour, the other at the

rate of 28 miles an hour. The slower of the two trains is 12 miles in advance of the other.

(1) How long before the faster train will be within 4 miles of the slower train?

(2) How long before it will be up with it?

(3) How long before it will be 6 miles in advance of it?

Let $x =$ the number of hours in case (1).

$30 - 28 =$ the number of miles gained in one hour.

Then $x(30 - 28) =$ " " " " x hours.

But $8 =$ " " " " "

$$\therefore \quad x(30 - 28) = 8$$

$$2x = 8$$

$x = 4$, the number of hours in case (1).

Let $x' =$ the number of hours in case (2);

then

$$x'(30 - 28) = 12$$

$$2x' = 12$$

$x' = 6$, the number of hours in case (2).

Let $x'' =$ the number of hours in case (3);

then

$$x''(30 - 28) = 18$$

$$2x'' = 18$$

$x'' = 9$, the number of hours in case (3).

It is left to the pupil to determine the *number of miles* the trains run in each case.

4. Suppose the trains in Ex. 3 were 174 miles apart, and running towards each other, how many hours before they would meet? How many miles would each train run?

Let $x =$ the number of hours.

$30 + 28 =$ the rate of approach in miles an hour.

Then $(30 + 28)x =$ the number of miles passed over in x hours.

But $174 =$ " " " " "

$$\therefore \quad (30 + 28)x = 174$$

$$58x = 174$$

$x = 3$, the number of hours.

The number of miles would be 90 and 84.

5. Suppose in Ex. 3 the tracks were circular, and 20 miles long. If the trains were side by side at 12 o'clock M., when would they next be in the same position?

Let $x =$ the number of hours,

and $30 - 28 =$ the number of miles gained in one hour;

then $(30 - 28)\,x =$ " " " " x hours.

But $20 =$ " " " " "

$$\therefore\ (30-28)\,x = 20$$

$$2\,x = 20$$

$x = 10$, the number of hours.

The time would be 10 o'clock, P. M.

6. Find the time between 4 and 5 o'clock, when the hands of a clock are,

(1) Together.
(2) Opposite each other.
(3) At right angles to each other.
(4) Two minute-spaces from each other.

In all these cases, let the position of the hands at 4 o'clock be their starting position. Then, in case (1), the minute-spaces to be gained are 20; in case (2), 50; in case (3), 5, or 35; in case (4), 18, or 22.

The rates an hour of the minute-hand and hour-hand are 60 minute-spaces, and 5 minute-spaces, respectively.

Let $x =$ the number of minutes past 4 in case (1).

Now $60 - 5 =$ the number of spaces gained in one hour,

and $\frac{60-5}{60} =$ " " " " a minute.

Then $\frac{60-5}{60}\,x = \frac{11\,x}{12} =$ the number of spaces gained in x minutes.

But $20 =$ " " " " "

$$\therefore\ \frac{11\,x}{12} = 20$$

$x = 21\frac{9}{11}$, the number of minutes.

Hence, the time is $21\frac{9}{11}$ minutes past 4 o'clock.

Let $x' =$ the number of minutes past 4 in case (2);

then $\frac{11\,x'}{12} =$ the number of spaces gained in x' minutes.

But $50 =$ the number of spaces gained in x' minutes;

$$\therefore \quad \frac{11x'}{12} = 50$$

$x' = 54\frac{6}{11}$, the number of minutes.

Hence, the time is $54\frac{6}{11}$ minutes past 4 o'clock, or $5\frac{5}{11}$ minutes of 5.

Let $x'' =$ the number of minutes past 4 in case (3).

Then $$\frac{11x''}{12} = 5, \text{ or } 35$$

$x'' = 5\frac{5}{11}$, or $38\frac{2}{11}$, the number of minutes.

Hence, the time is either $5\frac{5}{11}$ minutes or $38\frac{2}{11}$ minutes past 4 o'clock.

Let $x''' =$ the number of minutes past 4 in case (4).

Then $$\frac{11x'''}{12} = 18, \text{ or } 22$$

$x''' = 19\frac{7}{11}$, or 24, the number of minutes.

Hence, the time is either $19\frac{7}{11}$ minutes or 24 minutes past 4 o'clock.

7. A dog pursues a hare. The hare gets a start of 50 of her own leaps. The hare makes 6 leaps while the dog makes 5, and 7 of the dog's leaps are equal to 9 of the hare's. How many leaps will the hare take before she is caught? How many leaps will the dog take?

Let $\frac{1}{7}$ of the hare's leap be the *unit of measure*.

Then $7 =$ the length, in these units, of the hare's leap,

and $9 =$ " " " " dog's "

Since the hare takes 6 leaps to the dog's 5,

therefore $42 =$ the distance the hare goes in a certain unit of time,

and $45 =$ " " dog goes in the *same* " "

$\therefore \quad 45 - 42 =$ the gain of the dog in a unit of time.

Let $x =$ the units of time before the hare is caught;

then $(45 - 42)x =$ the distance the dog gains.

But $50 \times 7 =$ " " "

$\therefore \quad 3x = 350$

$x = 116\frac{2}{3}$, the units of time.

$\therefore \quad \dfrac{116\frac{2}{3} \times 42}{7} = 700$, the number of leaps the hare takes,

and $\dfrac{116\frac{2}{3} \times 45}{9} = 583\frac{1}{3}$, " " " dog "

8. A and B are talking of their ages; A says to B that if three fourths, three tenths, and four fifths of his age are added to his age, the sum will be 3 less than three times his age. What was A's age?

9. A grocer has some tea worth 40 cents a pound, and some worth 70 cents a pound. How many pounds must he take of each sort to produce 100 pounds of a mixture worth 50 cents a pound?

10. A person had $1000, of which he lent part at 4 per cent, and the rest at 5 per cent. The whole annual interest received was $44. How much was lent at 4 per cent?

Ans. $600.

11. An army in a defeat loses one sixth of its number in killed and wounded, and 4000 taken prisoners. It is then reinforced with 3000 men, but retreats, losing one fourth of its number in doing so. There remain 18000 men. What was the original number?

12. In a certain weight of gunpowder the saltpetre was 6 pounds more than half of the weight, the sulphur 5 pounds less than a third, and the charcoal 3 pounds less than a fourth. How many pounds were there of each of the three ingredients?

13. A general, having lost a battle, found that he had left fit for action 3600 men more than half of his army. 600 more than one eighth of his army were wounded, and the remainder, forming one fifth of the army, were slain, taken prisoners, or missing. What was the number of the army at first?

Ans. 24000.

14. A colonel, on attempting to draw up his regiment in the form of a solid square, finds that he has 31 men over, and that he should require 24 men more in his regiment in order to increase the side of the square by one man. How many men were there in the regiment?

15. Divide the number 90 into four parts such that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2 will all be equal.

16. A can do half as much work as B, B can do half as much as C, and together they can complete a piece of work in 24 days. In what time can each alone complete the work?

Ans. 168, 84, 42.

17. A regiment was drawn up in a solid square. After 295 men had been removed from the field, it was again drawn up in a solid square, and it was found that there were 5 men less in a side. What was the original number of men in the regiment? Ans. 1024.

18. A and B shoot at a target. A puts 7 bullets out of 12 into the bull's-eye, B 9 out of 12, and both of them put in 32 bullets. How many shots did each fire?

19. A mother is 70 years old, her daughter is exactly half that age. How many years have passed since the mother was $3\frac{1}{2}$ times the age of the daughter?

20. A basket of oranges is emptied by one person taking half of them and one more, a second person taking half of the remainder and one more, and a third taking half of the second remainder and six more. How many did the basket contain at first? Ans. 54.

21. A man leaves two fifths of his property to his eldest son, one half of the remainder to be divided equally between two younger sons, and the other half equally among three daughters. What is the value of the property, if the younger sons each get $1250 more than each of their sisters?

Ans. $25000.

22. A trader allows $100 for expenses, and increases that part of his capital which is not expended by $\frac{1}{3}$ of it. At the end of 3 years his stock is doubled. What had he at first?

23. A man meets 3 beggars. To the first he gives half of the coppers in his pocket and one more; to the second, half of the remainder and one more; and to the third half of what he has left, and one more. After this he finds he has but 3 coppers. How many had he at first? Ans. 38.

24. The length of a floor exceeds the breadth by 4 feet. If each dimension is increased a foot, the area of the room will be increased 27 square feet. Find the dimensions.

25. A and B have the same income. A saves a fifth of his, but B, by spending annually \$80 more than A, at the end of 4 years finds himself \$220 in debt. What was their income?

26. A can do half as much as B, B can do one and a half times as much as C, and together they can complete a piece of work in 10 days. How long would it take each separately to do the work?

27. A and B can dig a trench in 7 days, A and C in 8 days, and A alone in 15 days. How many days would it take A, B, and C together to dig it?

28. A can do a piece of work in 5 days, B can work twice as fast as A, and C one half as fast. How long would it take A, B, and C together to do the work?

29. A and B can do a piece of work together in 4 days. A working with a different companion, C, can do it in $3\frac{3}{5}$ days, while B and C working together can do it in $5\frac{1}{7}$ days. How many days will A, B, and C each take to do it alone?

Ans. A 6 days, B 12, C 9.

30. A tank is filled through a pipe in 30 minutes. It is emptied through a waste-pipe in 50 minutes. In what time will the tank be filled if both pipes are opened at once?

31. A cistern can be filled by means of two pipes in 20 minutes and 30 minutes, respectively, and emptied by means of a third in 40 minutes. In what time would it be filled if all three were running together?

32. A tank containing 800 gallons has three pipes. The first lets in 8 gallons in $2\frac{1}{2}$ minutes, the second 10 gallons in $3\frac{1}{3}$ minutes, and the third 12 gallons in 5 minutes. In what time will the tank be filled by the three pipes all running together?

33. A can do a piece of work in 5 days, B in 7 days, and C in 10 days. How long will it take them all working together to do it?

34. A can build a boat in 18 days, but with the assistance of B he can do it in 12 days. How long would it take B working alone to do it?

35. A can do a piece of work in 5 days, B in $3\frac{1}{3}$ days, and C in $5\frac{1}{2}$ days. How long would it require them working together to do it?

36. A performs $\frac{3}{7}$ of a piece of work in 15 days; he then calls in B to help him, and the two together finish the work in 8 days. In how many days can each alone do the work?

37. A can do in 20 days a piece of work which B can do in 30 days. A begins the work, but after a time B takes his place and finishes it. B worked 10 days longer than A. How long did A work? Ans. 8 days.

38. A deer running at the rate of 40 rods a minute was first seen by a huntsman when 30 rods in advance of a hound pursuing at the rate of 50 rods a minute. The huntsman waited until 25 rods intervened between the hound and the deer, when he shot the deer. How long did he wait?

39. There are two places 154 miles apart, from which two persons start at the same time toward each other. One travels at the rate of 3 miles in 2 hours, and the other at the rate of 5 miles in 4 hours. Where will they meet?

40. A privateer sailing at the rate of 10 miles an hour discovers a ship 18 miles off sailing at the rate of 8 miles an hour. How many miles can the ship sail before it is overtaken?

41. A freight train passes through a station at 20 miles an hour, and is followed at a distance of 2 miles by an express going 60 miles an hour. Where will the collision occur?

42. A courier sets out from a certain place and travels 17 miles in 4 hours, and $2\frac{1}{4}$ hours afterward a second courier, travelling 13 miles in 3 hours, is sent after him. How long and how far will the first go before being overtaken?

43. A person walked to the top of a mountain at the rate of $2\frac{1}{3}$ miles an hour, and down the same way at the rate of $3\frac{1}{2}$ miles an hour, and was out 5 hours. How far did he walk?

44. A hare takes 4 leaps to a greyhound's 3, but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 leaps. How many leaps must the greyhound take to catch the hare? Ans. 300.

45. A merchant bought two pieces of cloth, the first at the rate of \$4 for 9 yards, and the second at that of \$3 for 4 yards; the second piece contained as many times 3 yards as the first contained times 4 yards. He sold each piece at the rate of \$5 for 9 yards, and lost \$5 by the bargain. How many yards were there in each piece?

Ans. First, 144; second, 108.

46. At what time between 5 and 6 o'clock will the hands of a watch be together?

47. At what time between 7 and 8 o'clock will the hands of a watch be opposite one another?

48. At what times between 6 and 7 o'clock will the hands of a clock be at right angles?

49. A certain fraction is equal to $\frac{2}{3}$, and if its numerator is increased by 5 and its denominator by 9 it becomes $\frac{5}{8}$. Find the fraction.

50. Two laborers are employed at \$3 and \$5 a day each. The sum of the days they worked was 40. They each received the same sum. How many days was each employed?

CHAPTER XII.

EQUATIONS

OF THE FIRST DEGREE CONTAINING TWO OR MORE UNKNOWN NUMBERS.

176. Independent Equations are such as cannot be derived from one another, or reduced to the same form.

Thus, $x + y = 10$, $\frac{x}{2} + \frac{y}{2} = 5$, and $4x + 3y = 40 - y$, are not independent equations, since any one of the three can be derived from any other one; or they can all be reduced to the form $x + y = 10$. But $x + y = 10$ and $4x = y$ are independent equations.

177. Simultaneous Equations are those which are satisfied by the *same values* of the unknown numbers.

Thus, $x + y = 10$, and $x + y = 7$, though independent, are not simultaneous equations, since no values of x and y will satisfy both equations. But $x + y = 10$ and $4x = y$ are simultaneous, as well as independent equations.

178. Two independent simultaneous equations are necessary to determine the values of two unknown numbers.

For from the equation $x + y = 10$ we cannot determine the value of either x or y in known terms. If y is transposed, we have $x = 10 - y$; but since y is unknown, we have not determined the value of x. We may suppose y equal to any number whatever, and then x would equal the remainder obtained by subtracting y from 10. It is only required by the equation that the sum of two numbers shall equal 10; but there is an infinite number of pairs of numbers whose sum is equal to 10. But if we have also the equation $4x = y$, we may put this value of y in the first equation, $x + y = 10$, and obtain $x + 4x = 10$, or $x = 2$; then $4x = 8 = y$, and we have the value of each of the unknown numbers. But from the two equations $x + y = 10$ and $x + y = 7$ we can find no values of x and y that will satisfy both equations.

ELIMINATION.

179. Elimination is the method of deriving from the given equations a new equation, or equations, containing one (or more) less unknown number. The unknown number thus excluded is said to be *eliminated.*

There are three methods of elimination:

I. By substitution.
II. By comparison.
III. By combination.

Case I.

180. Elimination by Substitution.

1. Solve $\begin{cases} 3x + 7y = 27. & (1) \\ 5x + 2y = 16. & (2) \end{cases}$

$$y = \frac{27 - 3x}{7} \quad (3) \qquad 5x + 2\left(\frac{27 - 3x}{7}\right) = 16 \quad (4)$$

$$35x + 54 - 6x = 112 \quad (5)$$

$$x = 2 \quad (6)$$

$$y = \frac{27 - 6}{7} = 3 \quad (7)$$

Transposing $3x$ in (1) and dividing by 7, we have (3), which gives an expression for the value of y. Substituting this value of y in (2), we have (4), which contains but one unknown number; that is, y has been eliminated. Reducing (4) we obtain (6), or $x = 2$. Substituting this value of x in (3), we obtain (7), or $y = 3$. Hence, the following

Rule.

Find an expression for the value of one of the unknown numbers in one of the equations, and substitute this value for the same unknown number in the other equation.

Note. After eliminating, the resulting equation is reduced by the rule in Art. 163. The value of the unknown number thus found must be substituted in one of the equations containing the two unknown numbers, and this reduced by the rule in Art. 163.

Solve the following equations by substitution:

2. $\begin{cases} 3x = 7y. \\ 12y = 5x - 1. \end{cases}$

3. $\begin{cases} 3x + 4y = 10. \\ 4x + y = 9. \end{cases}$

4. $\begin{cases} 8x - y = 34. \\ x + 8y = 53. \end{cases}$

5. $\begin{cases} 2x - 3y = -14. \\ 5y - 4x = 26. \end{cases}$

6. $\begin{cases} 7y - 21 = 5x. \\ 21x - 9y = 75. \end{cases}$

CASE II.

181. Elimination by Comparison.

1. Solve $\begin{cases} 3x + 7y = 27. & (1) \\ 5x + 2y = 16. & (2) \end{cases}$

$$x = \frac{27 - 7y}{3} \quad (3) \qquad x = \frac{16 - 2y}{5} \quad (4)$$

$$\frac{27 - 7y}{3} = \frac{16 - 2y}{5} \quad (5)$$

$$135 - 35y = 48 - 6y \quad (6)$$

$$y = 3 \quad (7)$$

$$x = \frac{16 - 6}{5} = 2 \quad (8)$$

Finding an expression for the value of x from both (1) and (2), we have (3) and (4). Placing these two values of x equal to each other (Art. 36, Ax. 8), we form (5), which contains but one unknown number. Reducing (5) we obtain (7), or $y = 3$. Substituting this value of y in (4), we have (8), or $x = 2$. Hence the following

Rule.

Find an expression for the value of the same unknown number from each equation, and put these expressions equal to each other.

Solve the following equations by comparison:

2. $\begin{cases} 2x + 3y = 7. \\ 5x + 7y = 19. \end{cases}$

3. $\begin{cases} x + 8y = 17. \\ 7x - 3y = 1. \end{cases}$

4. $\begin{cases} 19x + 17y = 0. \\ 2x - y = 53. \end{cases}$

5. $\begin{cases} 14x - 3y = 39. \\ 6x + 17y = 35. \end{cases}$

CASE III.

182. Elimination by Combination.

1. Solve $\begin{cases} 3x + 7y = 27. & (1) \\ 5x + 2y = 16. & (2) \end{cases}$

$$15x + 35y = 135 \quad (3)$$
$$15x + 6y = 48 \quad (4)$$
$$29y = 87 \quad (5) \qquad 5x + 6 = 16 \quad (7)$$
$$y = 3 \quad (6) \qquad x = 2 \quad (8)$$

If we multiply (1) by 5 and (2) by 3, we have (3) and (4), in which the coefficients of x are equal; subtracting (4) from (3), we have (5), which contains but one unknown number. Reducing (5) we have (6), or $y = 3$; substituting this value of y in (2), we obtain (7), which reduced gives (8), or $x = 2$. Hence the following

Rule.

Multiply or divide the equations so that the coefficients of the unknown number to be eliminated shall become equal; then, if the signs of this number are alike in both, subtract one equation from the other; if unlike, add the two equations together.

NOTE. The least multiplier for each equation will be that which will make the coefficient of the unknown number to be eliminated the least common multiple of the two coefficients of this number in the given equations. It is always best to eliminate that unknown number whose coefficients can most easily be made equal.

Solve the following equations by combination:

2. $\begin{cases} 5x + 6y = 17. \\ 6x + 5y = 16. \end{cases}$

5. $\begin{cases} 21x - 50y = 60. \\ -27y + 28x = 199. \end{cases}$

3. $\begin{cases} 15x + 77y = 92. \\ 5x - 3y = 2. \end{cases}$

6. $\begin{cases} 28x - 23y = 33. \\ 63x - 25y = 101. \end{cases}$

4. $\begin{cases} 6y - 5x = 18. \\ 12x - 9y = 0. \end{cases}$

7. $\begin{cases} 5x + 3y = 68. \\ 2x + 5y = 69. \end{cases}$

183. Solve the following equations:

NOTE. Which of the three methods of elimination should be used depends upon the relations of the coefficients to each other. That one which will eliminate the number desired with the least work is the best.

1. $$\begin{cases} 5(x+2y)-(3x+11y)=14. & (1) \\ 7x-9y-3(x-4y)=38. & (2) \end{cases}$$

$$5x+10y-3x-11y=14 \qquad 7x-9y-3x+12y=38$$

$$2x-y=14 \quad (3) \qquad 4x+3y=38 \quad (4)$$

From (1) and (2) we derive (3) and (4); reducing (3) and (4) by any of the methods of elimination, we have $x=8$, and $y=2$.

2. $$\begin{cases} 3x-\dfrac{y-5}{7}=\dfrac{4x-3}{2}. & (1) \\ \dfrac{3y+4}{5}-\frac{1}{3}(2x-5)=y. & (2) \end{cases}$$

$$42x-2y+10=28x-21 \qquad 9y+12-10x+25=15y$$

$$14x-2y=-31 \quad (3) \qquad 10x+6y=37 \quad (4)$$

From (1) and (2) we obtain (3) and (4); reducing (3) and (4), we find $x=-\frac{14}{13}$, and $y=\frac{207}{26}$.

3. $$\begin{cases} 4x-\frac{1}{3}(y-3)=5x-3. \\ 2y+\frac{1}{3}(2x-5)=\dfrac{21y+37}{6}. \end{cases}$$

4. $$\begin{cases} \dfrac{x}{2}-\frac{1}{3}(y-2)-\frac{1}{4}(x-3)=0. \\ x-\frac{1}{2}(y-1)-\frac{1}{3}(x-2)=0. \end{cases}$$

5. $$\begin{cases} (x+1)(y+5)=(x+5)(y+1). \\ xy+x+y=(x+2)(y+2). \end{cases}$$

Ans. $x=y=-2$.

6. $\dfrac{x}{3}+\dfrac{y}{4}=3x-7y-37=0.$

7. $\dfrac{x+3}{5}=\dfrac{8-y}{4}=\dfrac{3(x+y)}{8}.$

8. $\dfrac{x+1}{10}=\dfrac{3y-5}{2}=\dfrac{x-y}{8}.$ Ans. $\begin{cases} x=19. \\ y=\ \ 3. \end{cases}$

9. $\dfrac{x}{13}-\dfrac{y}{7}=6x-10y-8=0.$

10. $\begin{cases} \frac{1}{2}(x+y)+\frac{1}{2}(x-y)=3\frac{1}{2}. \\ \frac{1}{2}(x+y)-\frac{1}{2}(x-y)=2\frac{5}{9}. \end{cases}$

11. $\begin{cases} 0.7x-0.02y=2. \\ 0.7x+0.02y=2.2. \end{cases}$ Ans. $\begin{cases} x=3. \\ y=5. \end{cases}$

12. $\begin{cases} 1.75x=20-0.625y. \\ 2x-1.75y=0.6x+7. \end{cases}$

13. $\begin{cases} 2x-\dfrac{y+3}{4}=7+\dfrac{3y-2x}{5}. \\ 4y-\dfrac{8-x}{3}=24\frac{1}{2}-\dfrac{2y+1}{2}. \end{cases}$

14. $\begin{cases} \dfrac{x+\frac{y}{2}-3}{x-5}+7=0. \\ \dfrac{3y-10(x-1)}{6}+\dfrac{x-y}{4}+1=0. \end{cases}$ Ans. $\begin{cases} x=\ \ 4. \\ y=12. \end{cases}$

15. $\begin{cases} x-\dfrac{2y-x}{23-x}=20-\dfrac{59-2x}{2}. \\ y+\dfrac{y-3}{x-18}=30-\dfrac{73-3y}{3}. \end{cases}$

16. $\begin{cases} \dfrac{x-y}{x+y}=\dfrac{1}{5}. \\ \dfrac{\frac{2x}{3}-\frac{5y}{12}}{\frac{7}{4}}-\dfrac{\frac{3x}{2}-\frac{y}{3}}{\frac{23}{2}}=2. \end{cases}$ Ans. $\begin{cases} x=18. \\ y=12. \end{cases}$

17. $\begin{cases} a x + b y = c. & (1) \\ a' x + b' y = c'. & (2) \end{cases}$

Multiplying (1) by b',	$a b' x + b b' y = b' c$	(3)
Multiplying (2) by b,	$a' b x + b b' y = b c'$	(4)
Subtracting (4) from (3),	$(a b' - a' b) x = b' c - b c'$	
	$\therefore \ x = \dfrac{b' c - b c'}{a b' - a' b}$	
Multiplying (1) by a',	$a a' x + a' b y = a' c$	(5)
Multiplying (2) by a,	$a a' x + a b' y = a c'$	(6)
Subtracting (6) from (5),	$(a' b - a b') y = a' c - a c'$	
	$\therefore \ y = \dfrac{a' c - a c'}{a' b - a b'}$	

18. $\begin{cases} a x = b y. \\ b x + a y = c. \end{cases}$

19. $\begin{cases} a x + b y = a^2. \\ b x + a y = b^2. \end{cases}$

20. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{1}{a b}. \\ \dfrac{x}{a'} - \dfrac{y}{b'} = \dfrac{1}{a' b'}. \end{cases}$ Ans. $\begin{cases} x = \dfrac{a + a'}{a' b + a b'}. \\ y = \dfrac{b' - b}{a' b + a b'}. \end{cases}$

21. $\begin{cases} \dfrac{3 x}{a} + \dfrac{2 y}{b} = 3. \\ \dfrac{9 x}{a} - \dfrac{6 y}{b} = 3. \end{cases}$

22. $\begin{cases} (a - b) x = (a + b) y. \\ x + y = c. \end{cases}$

23. $\begin{cases} q x - r b = p (a - y). \\ \dfrac{q x}{a} + r = p \left(1 + \dfrac{y}{b}\right). \end{cases}$ Ans. $\begin{cases} x = \dfrac{a p}{q}. \\ y = \dfrac{b r}{p}. \end{cases}$

24. $\begin{cases} (a - b) x + (a + b) y = 2 a^2 - 2 b^2. \\ (a + b) x - (a - b) y = 4 a b. \end{cases}$

Ans. $\begin{cases} x = a + b. \\ y = a - b. \end{cases}$

25. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 1. \\ \dfrac{x}{3 a} + \dfrac{y}{6 b} = \dfrac{2}{3}. \end{cases}$

26. $\begin{cases} \dfrac{x}{a} - \dfrac{y}{b} = 1. \\ \dfrac{x}{b} + \dfrac{y}{a} = \dfrac{a}{b}. \end{cases}$

27. $\begin{cases} \dfrac{x-a}{c-a}+\dfrac{y-b}{c-b}=1. \\ \dfrac{x+a}{c}+\dfrac{y-a}{a-b}=\dfrac{a}{c}. \end{cases}$ Ans. $\begin{cases} x=c. \\ y=b. \end{cases}$

28. $\begin{cases} bx+cy=a+b. \\ ax\left(\dfrac{1}{a-b}-\dfrac{1}{a+b}\right)+cy\left(\dfrac{1}{b-a}-\dfrac{1}{b+a}\right)=\dfrac{2a}{a+b}. \end{cases}$

29. $\begin{cases} (a-b)x+(a+b)y=2(a^2-b^2). \\ ax-by=a^2+b^2. \end{cases}$

Ans. $\begin{cases} x=a+b. \\ y=a-b. \end{cases}$

30. $\begin{cases} \dfrac{8}{x}-\dfrac{9}{y}=1. & (1) \\ \dfrac{10}{x}+\dfrac{6}{y}=7. & (2) \end{cases}$

Multiplying (1) by 2, $\dfrac{16}{x}-\dfrac{18}{y}=2$

Multiplying (2) by 3, $\dfrac{30}{x}+\dfrac{18}{y}=21$

Adding, $\dfrac{46}{x}=23$

Whence, $x=2$

Substituting in (1), $y=3$

31. $\begin{cases} \dfrac{5}{x}+\dfrac{6}{y}=3. \\ \dfrac{15}{x}+\dfrac{3}{y}=4. \end{cases}$

33. $\begin{cases} \dfrac{9}{x}-\dfrac{5}{y}=2. \\ \dfrac{5}{x}+\dfrac{3}{y}=30. \end{cases}$

32. $\begin{cases} \dfrac{5}{x}+\dfrac{16}{y}=79. \\ \dfrac{16}{x}-\dfrac{1}{y}=44. \end{cases}$

34. $\begin{cases} \dfrac{25}{x}+\dfrac{24}{y}=1. \\ 20\left(\dfrac{2}{x}+\dfrac{3}{y}\right)=7. \end{cases}$

35. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = m. \\ \dfrac{a}{x} - \dfrac{b}{y} = n. \end{cases}$ Ans. $\begin{cases} x = \dfrac{2a}{m+n}. \\ y = \dfrac{2b}{m-n}. \end{cases}$

36. $\begin{cases} \dfrac{2}{ax} + \dfrac{3}{by} = 5. \\ \dfrac{5}{ax} - \dfrac{2}{by} = 3. \end{cases}$ Ans. $\begin{cases} x = \dfrac{1}{a}. \\ y = \dfrac{1}{b}. \end{cases}$

37. $\begin{cases} \dfrac{2}{x} - \dfrac{5}{3y} = \dfrac{4}{27}. \\ \dfrac{1}{4x} + \dfrac{1}{y} = \dfrac{11}{72}. \end{cases}$ 38. $\begin{cases} \dfrac{m}{nx} + \dfrac{n}{my} = m + n. \\ \dfrac{n}{x} + \dfrac{m}{y} = m^2 + n^2. \end{cases}$

184. In order to solve equations which contain two unknown numbers we have seen (§ 178) that we must have two independent simultaneous equations. In like manner, to solve equations which contain more than two unknown numbers we must have as many independent simultaneous equations as there are unknown numbers.

1. Solve $\begin{cases} 6x + 2y - 5z = 13. \\ 3x + 3y - 2z = 13. \\ 7x + 5y - 3z = 26. \end{cases}$

$$\begin{array}{lr}
6x + 2y - 5z = 13 & (1) \\
6x + 6y - 4z = 26 & (4) \\
\hline
4y + z = 13 & (5) \\
 & \\
12 + z = 13 & (12) \\
z = 1 & (13)
\end{array}
\qquad
\begin{array}{lr}
3x + 3y - 2z = 13 & (2) \\
21x + 21y - 14z = 91 & (6) \\
21x + 15y - 9z = 78 & (7) \\
\hline
6y - 5z = 13 & (8) \\
20y + 5z = 65 & (9) \\
\hline
26y = 78 & (10) \\
y = 3 & (11)
\end{array}
\qquad
\begin{array}{lr}
7x + 5y - 3z = 26 & (3) \\
 & \\
 & \\
 & \\
 & \\
 & \\
7x + 15 - 3 = 26 & (14) \\
x = 2 & (15)
\end{array}$$

Multiplying (2) by 2, we obtain (4), from which we subtract (1) and obtain (5); multiplying (2) by 7 and (3) by 3, we obtain (6) and (7); subtracting (7) from (6), we obtain (8). We have now two equations, (5) and (8), containing but two unknown numbers. Multiplying (5) by 5, we obtain (9), which added to (8) gives (10), which reduced

gives $y = 3$. Substituting this value of y in (5) and reducing, we obtain $z = 1$. Substituting these values of y and z in (3), and reducing, we obtain $x = 2$.

Hence, for solving equations containing any number of unknown numbers, we have the following

Rule.

From the given equations deduce equations one less in number, containing one less unknown number; and continue thus to eliminate one unknown number after another, until one equation is obtained containing but one unknown number. Reduce this last equation so as to find the value of this unknown number; then substitute this value in an equation containing this and but one other unknown number, and, reducing the resulting equation, find the value of this second unknown number; substitute again these values in an equation containing no more than these two and one other unknown number, and reduce as before; and so continue, till the value of each unknown number is found.

NOTE. The process can often be very much abridged by the exercise of judgment in selecting the unknown number to be eliminated, the equations from which the other equations are to be deduced, the method of elimination which shall be used, and the simplest equations in which to substitute the values of the numbers which have been found.

Find the values of the unknown numbers in the following equations:

2. $\begin{cases} x + y + z + w = 19. \\ y + z + w + u = 22. \\ x + z + w + u = 21. \\ x + y + w + u = 20. \\ x + y + z + u = 18. \end{cases}$ Ans. $\begin{cases} x = 3. \\ y = 4. \\ z = 5. \\ u = 6. \\ w = 7. \end{cases}$

NOTE. If these equations are added together and the sum divided by 4, we shall have $x + y + z + w + u = 25$; and if from this the given equations are successively subtracted, the values of the unknown numbers become known.

3. $\begin{cases} 2x + 3y + 4z = 20. \\ 3x + 4y + 5z = 26. \\ 3x + 5y + 6z = 31. \end{cases}$

4. $\begin{cases} 4x - y - z = 5. \\ 3x - 4y + 16 = 6z. \\ 3y + 2(z-1) = x. \end{cases}$ Ans. $\begin{cases} x = 2. \\ y = -2. \\ z = 5. \end{cases}$

5. $\begin{cases} 7x + 3y - 2z = 16. \\ 2x + 5y + 3z = 39. \\ 5x - y + 5z = 31. \end{cases}$

6. $\begin{cases} 5x - 6y + 4z = 15. \\ 7x + 4y - 3z = 19. \\ 2x + y + 6z = 46. \end{cases}$ Ans. $\begin{cases} x = 3. \\ y = 4. \\ z = 6. \end{cases}$

7. $\begin{cases} 2x + 4y + 5z = 19. \\ -3x + 5y + 7z = 8. \\ 8x - 3y + 5z = 23. \end{cases}$

8. $\begin{cases} 5x + 6y - 12z = 5. \\ 2x - 2y - 6z = -1. \\ 4x - 5y + 3z = 7\frac{1}{2}. \end{cases}$ Ans. $\begin{cases} x = 2. \\ y = \frac{1}{2}. \\ z = \frac{2}{3}. \end{cases}$

9. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{5}z = 23. \\ \frac{1}{3}x + \frac{1}{3}y + \frac{1}{2}z = 27. \\ \frac{1}{5}x + \frac{1}{6}y + \frac{1}{3}z = 17. \end{cases}$ Ans. $\begin{cases} x = 30. \\ y = 6. \\ z = 30. \end{cases}$

10. $\begin{cases} y + z - 86 = 7z - 5x. \\ 93 - \frac{1}{2}z - \frac{1}{4}y = \frac{3}{4}y - 2z. \\ \frac{1}{4}x + \frac{1}{3}y + \frac{1}{2}z = 58. \end{cases}$

11. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 12 - \frac{1}{6}z. \\ \frac{1}{2}y + \frac{1}{3}z = 8 + \frac{1}{6}x. \\ \frac{1}{2}x + \frac{1}{3}z = 10. \end{cases}$

12. $\begin{cases} \frac{1}{4}(x-1) - \frac{1}{5}(y-2) = \frac{1}{10}(z+3). \\ x - \frac{1}{3}(2y-5) = 1\frac{3}{4} - \frac{1}{12}z. \\ y + \frac{1}{3}z = \frac{1}{5}x + 5. \end{cases}$

13. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5. \\ \dfrac{1}{y} + \dfrac{1}{z} = 8. \\ \dfrac{1}{x} + \dfrac{1}{z} = 7. \end{cases}$

NOTE. Subtract from half the sum of the three equations each equation successively.

14. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{4}. \\ \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{7}{36}. \\ \dfrac{1}{x} + \dfrac{1}{z} = \dfrac{5}{18}. \end{cases}$ Ans. $\begin{cases} x = 6. \\ y = 12. \\ z = 9. \end{cases}$

15. $\begin{cases} \dfrac{1}{x} - \dfrac{2}{y} + 4 = 0. \\ \dfrac{1}{y} - \dfrac{1}{z} + 1 = 0. \\ \dfrac{2}{z} + \dfrac{3}{x} = 14. \end{cases}$

16. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 36. \\ \dfrac{1}{x} + \dfrac{3}{y} - \dfrac{1}{z} = 28. \\ \dfrac{1}{x} + \dfrac{1}{3y} + \dfrac{1}{2z} = 20. \end{cases}$

17. $\begin{cases} \dfrac{2}{x} - \dfrac{5}{3y} + \dfrac{1}{z} = 3\dfrac{4}{27}. \\ \dfrac{1}{4x} + \dfrac{1}{y} + \dfrac{2}{z} = 6\dfrac{11}{72}. \\ \dfrac{5}{6x} - \dfrac{1}{y} + \dfrac{4}{z} = 12\dfrac{1}{36}. \end{cases}$

18. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = a. \\ \dfrac{1}{x} + \dfrac{1}{z} = b. \\ \dfrac{1}{y} + \dfrac{1}{z} = c. \end{cases}$ Ans. $\begin{cases} x = \dfrac{2}{a + b - c}. \\ y = \dfrac{2}{a - b + c}. \\ z = \dfrac{2}{b + c - a}. \end{cases}$

19. $\begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a. \\ \frac{1}{x} + \frac{1}{y} - \frac{1}{z} = b. \\ \frac{1}{x} - \frac{1}{y} + \frac{1}{z} = c. \end{cases}$

20. $\begin{cases} \frac{1}{x} - \frac{1}{y} - \frac{1}{z} = \frac{1}{a}. \\ \frac{1}{y} - \frac{1}{z} - \frac{1}{x} = \frac{1}{b}. \\ \frac{1}{z} - \frac{1}{x} - \frac{1}{y} = \frac{1}{c}. \end{cases}$

21. $\begin{cases} ly + mx = n. \\ nx + lz = m. \\ mz + ny = l. \end{cases}$

22. $\begin{cases} x + y + z = a + b + c. \\ 2x + 2y - 4z = 2c - a - b. \\ ax + by + cz = bc + ca + ab. \end{cases}$

Ans. $\begin{cases} x = \frac{1}{2}(b + c). \\ y = \frac{1}{2}(c + a). \\ z = \frac{1}{2}(a + b). \end{cases}$

23. $\begin{cases} x + y = 22. \\ y + z = 24. \\ z + w = 38. \\ w + u = 50. \\ u + x = 26. \end{cases}$

PROBLEMS

PRODUCING EQUATIONS OF THE FIRST DEGREE CONTAINING TWO OR MORE UNKNOWN NUMBERS.

185. Many of the problems given in Chapter XI. contain two or more unknown numbers; but in every case these are so related to one another that, if one becomes known, the others become known also; and therefore the problems can be solved by the use of a single letter. But many problems, on account of the complicated conditions, cannot be performed by the use of a single letter. No problem can be solved unless the conditions given are sufficient to form as many independent equations as there are unknown numbers.

1. Find two numbers such that the greater exceeds twice the less by 3, and twice the greater exceeds the less by 27.

Let $x =$ the greater number,

and $y =$ the less;

then, by the conditions, $x - 2y = 3$

and $2x - y = 27$

Solving these equations by either of the preceding methods, we have $x = 17$ and $y = 7$.

2. If the numerator of a fraction is increased by 2 and the denominator by 1, it becomes equal to $\frac{5}{8}$; and if the numerator and denominator are each diminished by 1, it becomes equal to $\frac{1}{2}$. Find the fraction.

Let $\frac{x}{y} =$ the fraction;

then, by the conditions, $\frac{x+2}{y+1} = \frac{5}{8}$ and $\frac{x-1}{y-1} = \frac{1}{2}$.

Solving these equations, we have $x = 8$ and $y = 15$, and the fraction is $\frac{8}{15}$.

3. A number consists of three digits whose sum is 10; the middle digit is equal to the sum of the other two, and, if 99 is added to the number, the order of the digits is reversed. Find the number.

Let $x =$ the digit in the units' place,

$y =$ " " tens' place,

$z =$ " " hundreds' place;

then $100z + 10y + x =$ the number.

By the conditions, $x + y + z = 10$

$$x + z = y$$

and $100z + 10y + x + 99 = 100x + 10y + z$

Solving these equations, we have $x = 3$, $y = 5$, $z = 2$.

$\therefore$ The number is 253.

4. A and B walk for a wager on a course of one mile in length. At the first heat, A gives B a start of 45 seconds,

and beats him by 110 feet. At the second heat, A gives B a start of 484 feet, and is beaten by 6 seconds. Required the rates at which A and B walk.

Let $x =$ the number of feet A walks a second.

$y =$ " " B " "

By the conditions of the problem,

$$\frac{5280}{x} + 45 = \frac{5170}{y} \qquad (1)$$

$$\frac{5280}{x} - 6 = \frac{4796}{y} \qquad (2)$$

Subtracting (2) from (1), $$51 = \frac{374}{y}$$

$$y = 7\tfrac{1}{3}$$

Substituting this value of y in (2), we find

$$x = 8$$

$\therefore$ A and B walk 8 and $7\frac{1}{3}$ feet per second, respectively.

5. I row 8 miles with the stream in 1 hour 4 minutes, and return against the stream in $2\frac{2}{7}$ hours. At what rate would I row in still water, and at what rate does the stream flow?

Let $x =$ number miles an hour in still water,

$y =$ number miles the stream flows an hour;

then, by the conditions of the problem,

$$x + y = \frac{8}{1\frac{1}{15}} = \frac{15}{2}$$

$$x - y = \frac{8}{2\frac{2}{7}} = \frac{7}{2}$$

Add $$2x = 11$$

$$x = 5\tfrac{1}{2}$$

Substituting this value of x in first equation, we find $y = 2$.

6. The denominator of a fraction exceeds the numerator by 4; and if 5 is taken from each, the sum of the reciprocal of the new fraction and 4 times the original fraction is 5. Find the original fraction.

7. A, B, C, and D, together, have \$270. A has three times as much as C, and B five times as much as D; and A and B together have \$50 less than eight times what C has. Find how much each has.

8. One eleventh of A's age is greater by two years than one seventh of B's, and twice B's age is equal to what A's age was thirteen years ago. Find their ages.

9. Find two numbers in the ratio of 3 to 4, such that, if the first is increased by 7 and the second doubled, these numbers will be in the ratio of 2 to 3.

10. A certain fraction becomes $\frac{1}{2}$ when its numerator is doubled and its denominator increased by 1; but if its numerator is increased by 1 and its denominator diminished by 1, it becomes $\frac{1}{3}$. Find the fraction.

11. A certain fraction is doubled by adding 14 to its numerator and 6 to its denominator, and it is trebled by adding 7 to its numerator and taking 4 from its denominator. Find the fraction. Ans. $\frac{7}{12}$.

12. If $\frac{2}{3}$ is added to the numerator of a certain fraction, the fraction will be increased by $\frac{1}{21}$; and if $\frac{1}{2}$ is taken from its denominator, the fraction becomes $\frac{2}{3}$. Find the fraction.

13. The middle digit of a number between 100 and 1000 is zero, and the sum of the other digits is 11. If the order of the digits is reversed, the number so formed exceeds the original by 495. Find the number. Ans. 308.

14. A number of 3 digits has the right-hand digit zero. If the left-hand and middle digits are interchanged, the number is diminished by 180; if the left-hand digit is halved and the middle and right-hand digits are interchanged, the number is diminished by 454. Find the number.

15. In a number of 4 digits the sum of the first and third is equal to the sum of the second and fourth; and, if the order of the digits is reversed, the number is increased by 1089. The difference between the third and fourth digits is three times the difference between the first and fourth, and the sum of the digits is 20. Find the number.

16. If I divide a certain number by the sum of its two digits, the quotient is 6 and the remainder 3. If I reverse the order of the digits and divide the resulting number by the sum of the digits, the quotient is 4 and the remainder 9. Find the number.

17. If I divide a certain number by the sum of its two digits diminished by 2, the quotient is 5 and the remainder 1. If I reverse the order of the digits and divide the resulting number by the sum of the digits increased by 2, the quotient is 5 and the remainder 8. Find the number. Ans. 36.

18. Having \$45 to distribute among a number of persons, I find that for a distribution of \$3 to each man and \$1 to each woman I shall lack \$1, but I can give \$$2\frac{1}{2}$ to every man and \$$1\frac{1}{2}$ to every woman, and have nothing left. How many men and women were there?

19. A man put \$12000 at interest in three sums, the first at 5 per cent, the second at 4 per cent, and the third at three per cent, receiving for the whole \$490 a year. The sum at 5 per cent is half as much as the other two sums. What are the three sums? Ans. \$4000, \$5000, \$3000.

20. As John and James were talking of their money, John said to James, "Give me 16 cents, and I shall have four times as much as you will have left." James said to John, "Give me 8 cents, and I shall have as much as you will have left." How many cents did each have?

Ans. John, 48 cents; James, 32 cents.

21. An income of \$120 a year is derived from a sum of money invested partly in $3\frac{1}{2}$ per cent stock, and partly in 4 per cent stock. If the stock should be sold out when the $3\frac{1}{2}$ per cents are at 108 and the 4 per cents at 120, the capital realized would be \$3672. How much stock of each kind is there? Ans. \$2400 $3\frac{1}{2}$ per cent, \$900 4 per cent.

22. A market-man bought eggs, some at 3 for 5 cents and some at 4 for 5 cents, and paid for the whole \$4.60; he afterward sold them at 24 cents a dozen, clearing \$0.80. How many of each kind did he buy?

23. A fishing-rod consists of two parts; the length of the upper part is to the length of the lower as 5 to 7; and 9 times the upper part together with 13 times the lower part exceeds 11 times the whole rod by 36 inches. Find the lengths of the two parts.

24. There is a rectangular floor, such that, if it were two feet broader and three feet longer, it would be sixty-four square feet larger; but if it were three feet broader and two feet longer, it would be sixty-eight square feet larger. Find the length and breadth of the floor. Ans. 14, 10.

25. A rectangle is of the same area as another which is 6 meters longer and 4 meters narrower; it is also of the same area as a third, which is 8 meters longer and 5 meters narrower. What is its area?

26. Find the length and breadth of a rectangle, such that, if 3 feet were taken from the length and added to the breadth, its area would be increased by 6 square feet; but if 3 feet were taken from its breadth and 4 feet added to its length, its area would be diminished by 20 square feet.

27. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards, but if the circumference of the fore-wheel were increased by a fourth of its present length, and that of the hind-wheel by a fifth of its present

length, the fore-wheel would make 4 revolutions more than the hind-wheel in going 120 yards. Find the circumferences of the two wheels. Ans. 4 yards and 5 yards.

28. A crew which can row 12 miles an hour down a river, finds that it takes twice as long to row up the river as to row down. At what rate does the water flow?

29. A man sculls for 1 hour and 40 minutes down a stream which runs at a rate of 4 miles an hour. In returning, it takes him 4 hours and 15 minutes to arrive at a point 3 miles short of his starting-place. Find the distance he sculled down the stream, and the rate of his sculling.

Ans. 20 miles; 8 miles an hour.

30. A person swimming in a stream which runs $1\frac{1}{2}$ miles an hour finds that it takes him 4 times as long to swim a mile up the stream as it does to swim the same distance down. At what rate does he swim?

31. A boat's crew row 9 miles with the tide in $\frac{3}{4}$ of an hour, and when the tide is flowing at half its former rate the same crew row 9 miles against the tide in $1\frac{1}{2}$ hours. Required the rate of the stronger tide, and the rate at which the crew can row in still water.

32. A and B can perform a piece of work together in 4 days, A and C in $3\frac{3}{5}$ days, and B and C in $5\frac{1}{7}$ days. Find the time in which each can perform the work alone.

33. A and B can together perform a certain work in 30 days; at the end of 18 days, however, B is called off, and A finishes it alone in 20 more days. Find the time in which each can perform the work alone.

34. A and B working together can do a piece of work in $2\frac{2}{5}$ days. It can also be done if A works 3 days and B 2 days. In what time can each of them do the work alone?

Ans. 6 days; 4 days.

35. A cistern has 3 pipes opening into it. If the first should be closed, the cistern would be filled in 10 minutes; if the second, in 15 minutes; and if the third, in 20 minutes. How long would it take each pipe alone to fill the cistern, and how long would it take the three together?

36. A and B together completed a piece of work in $2\frac{1}{12}$ days; but if A had worked one half as fast, and B twice as fast, they would have finished it in $3\frac{13}{19}$ days. In how many days could each alone perform the work?

37. 24 Ovids and 12 Cæsars will just fill a certain shelf. 6 Ovids and 10 Cæsars will fill half of it. How many of each alone will fill it?

38. A and B run a mile. At the first heat A gives B a start of 20 yards, and beats him by 30 seconds. At the second heat A gives B a start of 32 seconds, and beats him by $9\frac{5}{11}$ yards. Find the rate at which A runs.

39. A fox is pursued by a greyhound, and is 60 of her own leaps before him. The fox takes 3 leaps in the time that the greyhound takes 2; but the greyhound goes as far in 3 leaps as the fox does in 7. In how many leaps will the greyhound catch the fox?

40. Three men, A, B, and C, had together a certain sum of money. Now, if A gives to B and C as much as they already have, and then B gives to A and C as much as they have after the first distribution, and again C gives to A and B as much as they have after the second distribution, they will each have \$6. How much did each have at first?

Ans. A \$$9\frac{3}{4}$, B \$$5\frac{1}{4}$, and C \$3.

41. A and B run a mile. First, A gives B a start of 44 yards, and beats him by 51 seconds; at the second heat, A gives B a start of 1 minute 15 seconds, and is beaten by 88 yards. Find the time in which A and B each can run a mile.

42. Two baskets contain mixtures of wheat and barley. In the first there is 3 times as much wheat as barley, and in the second 5 times as much barley as wheat. Find how much must be taken from each to fill a third basket which holds 7 liters, in order that its contents may be half wheat and half barley.

Ans. 4 liters from the first, and 3 from the second.

43. There are two alloys of silver and copper, of which one contains twice as much copper as silver, and the other three times as much silver as copper. How much must be taken from each to weigh a kilogram, of which the silver and the copper shall be equal in quantity?

44. A and B are travelling on roads which cross each other. When B is at the point of crossing, A has 675 meters to go before he arrives at this point, and in 5 minutes they are equally distant from this point; and in 40 minutes more, they are again equally distant from it. What is the rate of each?

Ans. A's, 75; B's, 60 meters a minute.

45. A sets out from C to D. Three hours afterward B sets out from D to C, travelling 2 miles an hour more than A. When they meet, their distances are as 13 : 15. Now, had A travelled 5 hours less, and B 2 miles an hour faster, their distances would have been as 2 : 5. How many miles did each go an hour, and how many hours did each travel before meeting?

46. A and B, whose times to reap an acre are as 2 : 3, engaged to reap a field in 12 days. Finding themselves unable to finish it, they take in C the last few days. C's rate of working was such that, if he had worked with them from the beginning, they could have finished it in 9 days. And the time in which C could reap it with A and B severally is as 7 : 8. When was C called in?

Ans. C was called in after A and B had worked 6 days.

CHAPTER XIII.

GENERALIZATION.

NEGATIVE ANSWERS. DISCUSSION OF PROBLEMS.

186. SINCE letters stand for any numbers whatever, the answer to a problem in which the given numbers are represented by letters is a general expression including all cases of the same kind. Moreover, the operations performed with letters are not, as with figures, lost in the combinations of the process, but all appear in the resulting expression. Such an expression is called a *formula*, and when expressed in words a *rule*.

187. To illustrate the making of formulas, and their use, the following problems are solved.

1. If A can do a piece of work in a hours and B can do it in b hours, how long will it take A and B together to do the work?

Let $x =$ the required number of hours;

then $$\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$$

$$bx + ax = ab$$

$$x = \frac{ab}{a+b}, \text{ the required number of hours.}$$

That is, $\frac{ab}{a+b}$ is the formula that expresses, in terms of the time in which each can do it alone, the time it will take two men together to do a given piece of work. Stated in words, it will be as follows: Given the time it takes each of two men to do a piece of work, to find the time it will take the two to do it together,

Rule.

Divide the product by the sum of the numbers expressing the time it will take each to do it alone.

2. If A can do a piece of work in 5 days and B can do it in 7 days, how long will it take A and B together to do the work?

Ans. $\frac{5 \cdot 7}{5 + 7}$, or $2\frac{11}{12}$ days.

An infinite number of examples can be made by changing the numbers to represent a and b.

3. If A can do a piece of work in a days, B in b days, and C in c days, how long will it take A, B, and C together to do the work?

Let $x =$ the required number of days;

then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{x}$

$bcx + acx + abx = abc$

$x = \frac{abc}{ab + bc + ac}$, the required number of days.

That is, $\frac{abc}{ab + bc + ac}$ is the formula that expresses, in terms of the time it takes each to do the work, the time it will take three men together to do a piece of work. Stated in words, it will be as follows: Given the time it takes each of three men to do a piece of work, to find the time it will take the three to do it together,

Rule.

Divide the product by the sum of as many products as can be made by taking the three given numbers in pairs.

4. If A can do a piece of work in 3 hours, B in 4 hours, and C in 5 hours, how long will it take A, B, and C together to do the work?

Ans. $\frac{3 \cdot 4 \cdot 5}{3 \cdot 4 + 4 \cdot 5 + 3 \cdot 5}$, or $\frac{60}{12 + 20 + 15}$, or $1\frac{13}{47}$ hours.

Apply this Rule to Exs. 33, 35, p. 175.

5. Find the formula and state the rule when 4 men are employed.

6. Find the formula and state the rule when 5 men are employed.

7. Divide a into two parts such that the greater exceeds the less by b.

Let $x =$ the greater part;

then $x - b =$ the less part.

$$\therefore\ 2x - b = a$$

$$2x = a + b$$

$$x = \frac{a+b}{2}, \text{ the greater part;}$$

$$x - b = \frac{a-b}{2}, \text{ the less part.}$$

In this example a represents the sum and b the difference of two numbers, and $\frac{a+b}{2}$. $\frac{a-b}{2}$, are respectively the formulas for the two numbers. Stated in words, it will be as follows: Given the sum and difference of two numbers, to find the numbers,

Rule.

(1) *Divide the sum plus the difference of the two numbers by two, and it will give the greater.*

(2) *Divide the sum minus the difference of the two numbers by two, and it will give the less.*

8. By provision of the father's will, the sum of $5500 is to be divided between his two sons so that the elder shall receive $500 more than the younger. Find the share of each.

9. A and B together have 730 sheep, and A has 100 more than B. How many has each?

10. The sum of two numbers is 13.5, and their difference is 1. Find the numbers.

11. The sum of two numbers is $2a$, and their difference is $2b$. Find the numbers.

12. Three men, A, B, and C, form a partnership, and advance money in sums represented by p, p', p'', respectively. They gain a certain number of dollars, which we will represent by g. How ought the gain to be divided?

Let $px =$ the number of dollars in A's share,

then $p'x =$ " " " B's "

and $p''x =$ " " " C's "

$$\therefore\ px + p'x + p''x = g$$

$$x = \frac{g}{p + p' + p''}$$

Whence, $\dfrac{gp}{p + p' + p''} =$ A's share,

$$\frac{gp'}{p + p' + p''} = \text{B's} \quad "$$

$$\frac{gp''}{p + p' + p''} = \text{C's} \quad "$$

13. If each of the sums named in Example 12 is invested a certain time, say t, t', t'', respectively, how ought the gain to be divided?

In this case ptx, $p't'x$, $p''t''x$, would represent respectively the number of dollars in the share of each. That is,

$$ptx + p't'x + p''t''x = g$$

$$x = \frac{g}{pt + p't' + p''t''}$$

$$\therefore\ \frac{gpt}{pt + p't' + p''t''} = \text{A's share,}$$

$$\frac{gp't'}{pt + p't' + p''t''} = \text{B's} \quad "$$

$$\frac{gp''t''}{pt + p't' + p''t''} = \text{C's} \quad "$$

14. A, B, and C form a partnership. A puts in \$4000, B \$5000, C \$6000. They gain \$3000. How ought the gain to be divided?

15. Divide \$200 between John, Charles, and William, so that John shall have 2 dollars as often as Charles has 3 and William 5.

16. A, B, and C form a partnership. A furnishes \$700 for 11 months, B \$1100 for 8 months, and C \$900 for 12 months. They gain \$1365. What is each man's share of the gain?

17. A, B, and C bought a horse for \$200, and sold him for \$250, by which A gained \$25 and B \$10. How much had A, B, and C each paid for the horse?

18. What is the interest of p dollars for t years at r per cent? What the amount?

Let i represent the interest, and a the amount.

The interest = principal $\times$ time $\times$ rate.

$$\therefore \quad i = p\,t\,r, \text{ and } a = p + p\,t\,r$$

These formulas contain four different things, any one of which may be determined when the others are known. Deducing, for example, the value of t, we have

$$t = \frac{i}{p\,r}, \text{ or } \frac{a-p}{p\,r}.$$

19. At what rate must \$300 be put on interest to gain \$18 in 2 years?

20. What principal at 6% will amount to \$130.39 in 8 months?

21. How long must \$254 be on interest at 5% to gain \$44.45?

22. How long will it take any sum of money to double itself on interest at 5%?

23. How long will it take any sum to quadruple itself on interest at 10%?

24. What is the amount of p dollars, at compound interest, for n years, at r per cent?

The amount of one dollar, at compound interest, for one year, will be represented by $1 + r$; that of p dollars will be therefore $p(1+r)$. For the second year $p(1+r)$ will be the principal, and its amount will be $p(1+r)(1+r)$, or $p(1+r)^2$; for the third year $p(1+r)^3$, and so on. Therefore, putting a for the amount for n years, we have, $a = p(1+r)^n$.

25. What is the amount of \$200 for 4 years, at compound interest, at 5% ?

Ans. $a = p\,(1 + r)^n = \$200\,(1.05)^4 = \$243.10+.$

26. What principal, at compound interest at 6%, will amount to \$357.3048 in 3 years?

27. Find the amount of \$300, at compound interest, for 2 years 6 months, at 8%, the interest being compounded semiannually.

188. MISCELLANEOUS EXAMPLES.

1. Two persons have together \$$a$. One has m times as much as the other. How much has each?

2. If \$$a$ are divided among three persons, A, B, and C, so that B has \$$p$ more than A, and C \$$q$ more than B, how much has each?

3. If the mth and nth parts of a sum of money amount to \$$a$, what is the sum?

4. A certain number, a, is to be divided into two parts so that the mth part of the first, together with the nth part of the second, is equal to b. What are the parts?

Ans. $\dfrac{m}{n-m}(nb-a),\quad \dfrac{n}{m-n}(mb-a).$

5. A man spends the mth part of his income on board and lodging, the nth part on clothes, the pth part on amusements, the qth part in charities, and at the end of the year finds that he has \$$a$ left. What was his income?

6. The number a is divided into two parts, so that the $\frac{1}{m}$ of the first exceeds $\frac{1}{n}$ of the second by b. Find the parts.

7. If \$$s$ are divided among A, B, and C so that B has \$$a$ more than A, and C \$$b$ more than B, how much has each?

Ans. A \$$\frac{1}{3}(s-2a-b)$, B \$$\frac{1}{3}(s+a-b)$, C \$$\frac{1}{3}(s+a+2b)$.

8. If A can do a piece of work in a hours, B in b hours, and C in c hours, how long will they take working all together?

9. I row a miles down stream in b minutes, and return in c minutes. Find the rate at which I row in still water, and the rate at which the stream flows.

10. A has to write m lines, which he can do in p minutes. If B helps him they can do the work together in $p - q$ minutes. How long would it take B to do the work?

Ans. $\frac{p}{q}(p - q)$ min.

11. How many pounds of tea at m cents a pound must be mixed with p pounds at n cents a pound, that the mixture may be sold at s cents a pound?

12. A courier, A, travelling p miles in q hours, is followed at an interval of m hours by another, B, travelling r miles in s hours. How many hours will elapse before B overtakes A?

Ans. $\frac{mps}{qr - ps}$.

13. A and B can do a piece of work in a hours, B and C in c hours, A and C in b hours. How long will each take separately?

14. Two men set out at the same time to walk, one from A to B, and the other from B to A, a distance of a miles. The former walks at the rate of p miles, and the latter at the rate of q miles an hour. At what distance from A will they meet?

15. A man spends c dollars in buying two kinds of silk, at a dollars and b dollars a yard, respectively. He could buy three times as much of the first and half as much of the second for the same money. How many yards of each does he buy?

Ans. $\frac{c}{5a}$, $\frac{4c}{5b}$ yards.

16. A man rides one third of the distance from A to B at the rate of a miles an hour, and the remainder at the rate of $2b$ miles an hour. If he had travelled at a uniform rate of

$3c$ miles an hour, he could have ridden from A to B and back again in the same time. Prove that $\frac{2}{c} = \frac{1}{a} + \frac{1}{b}$.

17. Divide the number a into three parts, so that the first is to the second as $m : n$, and the second to the third as $p : q$.

18. A, B, and C start at the same time for a town p miles distant. A walks at a uniform rate of m miles an hour, and B and C drive at a uniform rate of n miles an hour. After a time B gets down and walks forward at the same rate as A, while C returns to meet A. A mounts with C, and they enter the town at the same time as B, C driving uniformly throughout. Show that the time of the journey is $\frac{p}{n} \cdot \frac{3n+m}{3m+n}$ hours.

INTERPRETATION OF NEGATIVE RESULTS AND OF THE FORMS $\frac{0}{A}$, $\frac{A}{0}$, $\frac{0}{0}$.

189. To explain negative results and the forms $\frac{0}{A}$, $\frac{A}{0}$, and $\frac{0}{0}$, we work the following problems.

1. A man and his son are employed to build a wall. At one time the father worked 7 days, and his son was with him 5 days, and for this time the payment was \$16.75. At another time the father worked 5 days, and his son was with him 3 days, and for this time the payment was \$12.25. What were the daily wages of the father and of the son?

Ans. Father's, \$2.75; son's, $-$\$0.50.

The negative answer for the son's wages shows that the son, instead of assisting the father, was a hindrance to him, and caused the father a loss of \$0.50 every day the son was with him. Is it right to say that the son is "employed," and to speak of the son's "wages"? If the example stated that the son did no work, but every day he was with the father he had to pay \$0.50 a day for his board, the answers would both be positive.

2. A is 40 years old, and twice as old as B. How many years hence will A be three times as old as B?

Ans. -10 years.

-10 years "*hence*" means 10 years *ago* (§ 38). Change "hence will A be" to *ago was A*, and the answer will be positive. The negative answer shows that one cannot be *three* times as old as another at a point of time *after* he is *twice* as old. *Arithmetically* the problem is impossible (§ 39, last paragraph).

Negative answers to problems can usually be interpreted, and the problem can generally be worded so as to change the negative to positive answers.

3. M and N are points on a straight road, m miles apart. A starts from M toward N at a miles an hour, and, h hours after A starts, B starts from N at b miles an hour. How far from N will A and B be together?

Let M R represent the road; M, A's starting point; N, B's starting point; and R, the point where A and B are together.

Let $\qquad x =$ the number of miles from N to R.

Then $\qquad \dfrac{x+m}{a} =$ the number of hours A travels,

and $\qquad \dfrac{x}{b} =$ the number of hours B travels.

Then
$$\frac{x+m}{a} = \frac{x}{b} + h$$
$$bx + bm = ax + abh$$
$$ax - bx = bm - abh$$
$$x = \frac{b(m - ah)}{a - b}, \text{ N to R.}$$

(1) Let $a > b$, and $m > ah$.

Then x, or N R, is positive, and R, the point where A and B are together (in the positive direction, § 39) is at the right of N. This is as it should be; for since $m > ah$, B is at the right of A when B starts, and, as $a > b$, A travels faster than B, and will overtake B somewhere at the right of N.

(2) Let $a < b$, and $m < a h$.

Then x, or N R, is positive, and R, the point where A and B are together, is at the right of N. This is as it should be; for since $m < a h$, A has already passed N when B starts, and A is at the right, or ahead of B, when B starts, and, as $a < b$, B travels faster than A, and will overtake A somewhere at the right of N.

(3) Let $a < b$, and $m > a h$.

Then x, or N R, is negative, and R, the point of meeting, if it take place, must be (in the negative direction) at the left of N. This is as it should be; for since $m > a h$, B is at the right of A when B starts, and, as $a < b$, A travels slower than B, and can never overtake B. To make the problem possible under this hypothesis, we must suppose that, at the moment of B's starting, A turns round and walks in the opposite direction, and B follows him toward M. Then R, where they are together, will be at the left of N.

(4) Let $a > b$, and $m < a h$.

Then x, or N R, is negative, and R, the point of meeting, if it take place, is at the left of N. This is as it should be; for, as in (2), A has passed N when B starts, and, as A travels faster than B, B can never overtake A. To make the problem possible, when B starts, both A and B, as in (3), must walk toward M.

(5) Let $a > b$, or $a < b$, and $m = a h$.

Then $$x\text{, or N R, } = \frac{0}{a - b} = 0.$$

This is as it should be; for since $m = a h$, A and B are together at N, and, since they travel at different rates, they will never be together again; therefore R, the place of meeting, is at N, and nowhere else.

(6) Let $a = b$, and $m > a h$, or $m < a h$.

Then $$x\text{, or N R, } = \frac{b\,(m - a h)}{0}.$$

Such a result may be regarded as a sign of impossibility; for whatever time we allow the two travellers, they can never come together, since, being once separated a given distance, and travelling equally fast, that distance will always be preserved. Nevertheless, such expressions as $\frac{A}{0}$ are considered by mathematicians as a value, to which they give the name of *infinity*. For if we suppose $a - b$ to be very small,

say one inch an hour, in the course of many miles A and B would come together, and if this small gain were divided by a number as large as we can imagine, R would be at an infinite distance from M. Hence we say $\frac{A}{0} = \infty$, which means that, by taking the denominator as small as we like, we can make the result as great as we can conceive.

(7) Let $a = b$, and $m = a h$.

Then $$x = \frac{0}{0}.$$

Since $m = a h$, A and B are together at N, and since $a = b$, they are always together; that is, R is at any point on the line N R, and there are as many solutions as one pleases. Therefore, the expression $\frac{0}{0}$ is called a *symbol of indetermination.*

EXAMPLES FOR PRACTICE.

190. Solve the following problems, and write each problem in a form to make it a possible arithmetical problem.

1. A, who is 30 years old, is three times as old as B. How many years hence will he be four times as old?

2. What two numbers are those whose difference is 48, and sum 22?

3. A cistern is being filled by three pipes. One pipe alone would fill it in 20 minutes, another in 30 minutes, and the cistern is filled in 40 minutes. In how many minutes would the third pipe fill it?

4. A starts upon a walk at the rate of 4 miles an hour, and after 15 minutes B starts from the same place at the rate of $3\frac{3}{4}$ miles an hour. Where will B overtake A?

5. A and B have \$8, A and C \$10, and B and C have \$35. How many has each?

6. A garrison of 1000 men was victualled for 30 days. After 10 days it was reinforced, and then the provisions lasted 25 days more. How many men were there in the reinforcement?

7. A man invested \$3000 in 3 per cents at 98, and on their rising sold out, and reinvested the proceeds in 4 per cents at 120, but did not change his income. Find the amount of the rise.

8. I began business with \$$a$, but having lost a certain sum I find that b times what I have left is equal to c times what I had at first. How much did I lose?

If $b = 3$ and $c = 4$, what modification must be made in the question to make it possible?

9. I began business with \$$a$, but having lost a certain sum I find that b times what I have left is the difference between c times what I had at first and d times what I lost. How many dollars did I lose?

Examine and interpret the results:

(1) When $d = 4,\ b = 7,\ c = 8.$
(2) " $d = 4,\ b = 7,\ c = 6.$
(3) " $d = 4,\ b = 4,\ c = 6.$
(4) " $d = b = c = 4.$

10. A cistern whose capacity is d gallons is filled by two pipes, A and B, and emptied by a waste-pipe, C, through which there flow a, b, and c gallons a minute, respectively. If A is opened for p minutes, and then closed, and then B and C are opened simultaneously, in how many minutes after B and C are opened will the cistern be full?

Let $x =$ the number of minutes.

Then $(b - c)\,x = d - ap$

$$x = \frac{d - ap}{b - c} \text{ the number of minutes.}$$

Interpret the results :

(1) If $b = 5,\ c = 6$.
(2) " $b = 4,\ c = 3,\ d = 40,\ a = 5,\ p = 8$.
(3) " $b = 6,\ c = 6,\ d = 40,\ a = 5,\ p = 7$.
(4) " $b = 6,\ c = 6,\ d = 40,\ a = 5,\ p = 8$.

(1) $$x = \frac{d - ap}{5 - 6},\ \text{a negative result.}$$

That is, the discharge pipe c being the greater, more runs out than in. The question should then read, In how many minutes will the cistern be *emptied?*

(2) $$x = \frac{40 - 5 \cdot 8}{4 - 3} = \frac{0}{1} = 0.$$

That is, the cistern is full at the moment A is closed.

(3) $$x = \frac{40 - 5 \cdot 7}{6 - 6} = \frac{5}{0}.$$

That is, when A is closed, the cistern lacks 5 gallons of being full, and the number of gallons in the cistern will neither be increased nor diminished after A is closed. But if c is very little less than b, say one drop, then in the course of millions of minutes the cistern would be full. By decreasing the size of the drop, we could increase the number of minutes to as many as we please ; we therefore consider the answer infinitely great, or infinity.

(4) $$x = \frac{40 - 5 \cdot 8}{6 - 6} = \frac{0}{0}.$$

That is, when A is closed, the cistern is full, and will always remain full, since just as much flows in as flows out. The value of x is therefore indeterminate.

11. A hound pursues a hare, and makes b leaps while the hare makes c leaps; but d hound-leaps equal e hare-leaps. The hare has a start of a leaps; when will the hound overtake the hare?

Interpret the results :

(1) When $a = 50,\ b = 7,\ c = 9,\ d = 5,\ e = 6$.
(2) " $a = 50,\ b = 9,\ c = 7,\ d = 6,\ e = 5$.
(3) " $a = 0,\ b = 7,\ c = 9,\ d = 5,\ e = 6$.
(4) " $a = 50,\ b = 6,\ c = 8,\ d = 3,\ e = 4$.
(5) " $a = 0,\ b = 6,\ c = 8,\ d = 3,\ e = 4$.

CHAPTER XIV.

INDETERMINATE EQUATIONS.

INEQUALITIES.*

191. IN order to solve equations containing unknown numbers, there must be as many equations as there are unknown numbers (§§ 178, 184). But with a less number of equations than there are unknown numbers, there may be conditions, or limitations, that determine what the solution or solutions must be.

Thus, suppose it is required to find the positive integral values of x and y in the equation $4x + 3y = 10$. If $x = 0$, y is not integral; if $x = 1$, $y = 2$; if $x = 2$, y is not integral; if $x > 2$, y is negative; therefore, the only solution with the conditions given is, $x = 1$, $y = 2$.

192. Equations containing more unknown numbers than there are equations are called **Indeterminate Equations.**

Solve in positive integers:

1. $3x + 5y = 20.$

$$
\begin{array}{llll}
\text{If} & x = 0, & y = 4 & (1) \\
\text{if} & x = 1, & y = & (2) \\
\text{if} & x = 2, & y = & (3) \\
\text{if} & x = 3, & y = & (4) \\
\text{if} & x = 4, & y = & (5) \\
\text{if} & x = 5, & y = 1 & (6) \\
\text{if} & x = 6, & y = & (7)
\end{array}
$$

In equations (2), (3), (4), (5), (7), as it can be seen at once that the value of y is a fraction, its value is not found. So it is clear that, if $x > 6$, y must be negative. Hence, the only possible answers are, $x = 0$, $y = 4$; or $x = 5$, $y = 1$.

* See Preface.

2. $7x + 2y = 18.$

3. $5x + 4y = 38.$

4. $3x + 11y = 42.$

5. $6x + 5y = 48.$

6. $3x + 5y = 16.$

$$3x = 16 - 5y$$
$$x = 5 - y + \frac{1-2y}{3}$$

Now as x, 5, and y are integers, $\frac{1-2y}{3}$ is an integer.

Let $$\frac{1-2y}{3} = m$$

Then $$1 - 2y = 3m$$

and $$2y = 1 - 3m$$
$$y = \frac{1-m}{2} - m$$

As before, $\frac{1-m}{2}$ must be an integer.

Let $$\frac{1-m}{2} = n$$

Then $$1 - m = 2n$$

and $$m = 1 - 2n$$
$$y = \frac{1-(1-2n)}{2} - (1-2n)$$
$$= 3n - 1$$
$$x = 5 - (3n-1) + \frac{1-2(3n-1)}{3}$$
$$= 7 - 5n$$

From $y = 3n - 1$, it is evident that, in relation to y, n must be positive, and not 0; while from $x = 7 - 5n$, it is evident that, in relation to x, n cannot have a positive value greater than 1. Hence n must equal 1.

Ans. $\begin{cases} x = 2. \\ y = 2. \end{cases}$

NOTE 1. It is better to find, in terms of the other, the value of the unknown number that has the smaller coefficient.

7. $5x - 8y = 7.$

$$x = y + 1 + \frac{3y + 2}{5}$$

Let $\qquad \dfrac{3y+2}{5} = m$

Then $\qquad 3y + 2 = 5m$

and $\qquad y = m + \dfrac{2(m-1)}{3}$

Then $\dfrac{m-1}{3}$ must be an integer.

Let $\qquad \dfrac{m-1}{3} = n$

Then $\qquad m - 1 = 3n$

$$m = 3n + 1$$

$$y = 3n + 1 + \frac{2(3n + 1 - 1)}{2} = 5n + 1$$

$$x = 5n + 1 + 1 + \frac{3(5n + 1) + 2}{5} = 8n + 3$$

If	$n = 0,$	$x = 3,$	$y = 1$
if	$n = 1,$	$x = 11,$	$y = 6$
if	$n = 2,$	$x = 19,$	$y = 11$
	&c.	&c.	&c.

NOTE 2. From Exs. 6 and 7 it will be seen that, if x and y are in the same member of the equation and have like signs, the number of positive integral solutions is limited; but if unlike signs, the number of solutions is infinite.

8. $4x + 9y = 75.$ 9. $7x - 4y = 53.$

10. $\begin{cases} 3x + 5y + z = 25. \\ 5x + 4y + 3z = 34. \end{cases}$

Ans. $x = 2,\ y = 3,\ z = 4.$

NOTE 3. Eliminate one of the unknown numbers; then proceed as before.

11. In how many ways can \$7 be paid in two-dollar bills *and* fifty-cent pieces?

Let $\qquad x =$ number of two-dollar bills;

$\qquad\qquad y =$ number of fifty-cent pieces.

Then
$$2x + \frac{y}{2} = 7$$
$$4x + y = 14$$

If	$x = 0,$	$y = 14$
if	$x = 1,$	$y = 10$
if	$x = 2,$	$y = 16$
if	$x = 3,$	$y = 2$

Ans. 3 ways.

12. In how many ways can \$0.50 be paid in three-cent and five-cent pieces?

13. In how many ways can \$27 be paid in five-dollar bills and two-dollar bills?

14. A owes B \$8.25. If A has only fifty-cent pieces and B only three-cent pieces, what is the simplest way for them to square accounts?

15. Divide 55 into two parts, so that one shall be divisible by 2 and the other by 3.

16. A drover buys sheep, turkeys, and hens. The whole number is 100, and the whole price \$100. For the sheep he pays \$3.50, for the turkeys, \$1.33$\frac{1}{3}$, and for the hens \$0.50 each. How many of each does he buy?

INEQUALITIES.

193. A statement that one number is greater or less than another (§ 26) is an *inequality*.

Thus, $7 > 5$ and $a < b$ are *inequalities*.

194. Inequalities of the *same direction* are those in which the sign $>$ (or $<$) points the same way in both; otherwise, the inequality is *reverse*.

Thus, $7 > 5$ and $a > b$ are of the *same direction*; but $7 > 5$ and $a < b$ are *reverse*.

195. *If equals are added to, or subtracted from, the members of an inequality, the inequality remains the same.*

Thus, if c is added to both members of $a > b$, it is evident that $a + c$ is as much greater than $b + c$ as a is greater than b. So $a - c$ is as much greater than $b - c$ as a is greater than b.

From this it follows that, as in equations, *a term can be transposed from one member of an inequality to the other, provided its sign is changed, without changing the inequality.* It also follows that, *if the signs of the terms of an inequality are changed, the inequality is reversed.*

Thus, if $a > b$, then $-a < -b$ (§ 39).

196. *If each member of an inequality is multiplied or divided by the same positive number, the inequality will be in the same direction as before.*

Thus, if $a > b$, then $10a > 10b$.

But if the multiplier is negative, the inequality is reversed.

Thus, if $a > b$, then $-10a < -10b$.

197. *Like powers of both members of an inequality, with both members positive, are inequalities of the same direction; but if both members are negative, the odd powers of the inequality are of the same, and the even powers of the reverse direction.*

Thus, if $a > b$, then $a^n > b^n$; but if $-a > -b$, then $(-a)^n > (-b)^n$ if n is odd, but $(-a)^n < (-b)^n$ if n is even.

198. *Like roots of both members of an inequality are inequalities of the same direction.*

Thus, if $a > b$, then $a^{\frac{1}{n}} > b^{\frac{1}{n}}$.

In case n is even, the inequality of the possible negative roots is reversed.

199. *The sum of the corresponding members of several inequalities of the same direction is an inequality of the same direction.*

Thus,

$$\begin{array}{r} 7 > 5 \\ 3 > 2 \\ 6 > 5 \\ \hline 16 > 12 \end{array}$$

200. *If one inequality is subtracted from another of the same direction the remainder is not necessarily an inequality of the same direction.*

Thus,

$$\begin{array}{c} 8 > 4 \\ 5 > 3 \\ \hline 3 > 1 \end{array} \qquad \begin{array}{c} 8 > 4 \\ 6 > 2 \\ \hline 2 = 2 \end{array} \qquad \begin{array}{c} 8 > 4 \\ 7 > 1 \\ \hline 1 < 3 \end{array}$$

201. These principles enable us to reduce an inequality so that the unknown number may stand alone as one member of the inequality.

Reduce:

1. $\frac{7x}{3} + \frac{1}{6} > 2x + \frac{5}{12}$.

Transposing and uniting, $\frac{x}{3} > \frac{1}{4}$

$$x > \tfrac{3}{4}$$

2. $5x - 8 < 3x - 5$. Ans. $x < \frac{3}{2}$.

3. $42x - 11x + 100 > 121 - 13x$.

4. $2x^2 - (x+1)^2 > (x+3)^2$.

5. $4\{(2x-1) - 2(x+1)\} < 3(x+5)$.

6. $5(x+1)^2 > 12(x+2)^2 - 7(x+3)^2$.

7. Given $\left\{\begin{array}{l} 3(x-5) - 8 > 2(x-4) \\ 5(x+3) + 2 < 3(x+17) \end{array}\right\}$ to find the limits of x.

Ans. $\left\{\begin{array}{l} x > 15. \\ x < 17. \end{array}\right.$

8. I paid for a horse an even number of dollars. If three times the price of the horse plus \$20 is more than twice the price plus \$169, and four times the price of the horse minus \$67 is less than three times the price plus \$84, what did I pay for the horse? Ans. \$150.

9. What number is that whose third minus its fifth is greater than 2, while its half plus its sixth is less than 12?

202. The exercises given below depend upon the following proposition:

If $a - b = \pm d$

then $a^2 - 2ab + b^2 = d^2$

and $a^2 + b^2 = 2ab + d^2$

If $\pm d = 0$, that is, if $a = b$, then $a^2 + b^2 = 2ab$; but if $\pm d$ does not equal 0, that is, if a does not equal b, then $a^2 + b^2 > 2ab$.

If the letters used below are positive and unequal, prove that

1. $a^3 - b^3 > 3a^2b - 3ab^2$. (Divide by $a - b$.)

2. $a^3 + b^3 > a^2b + ab^2$.

3. $(a + b - c)^2 + (a - b + c)^2 + (-a + b + c)^2 > ab + ac + bc$.

4. $\dfrac{a}{\sqrt{b}} + \dfrac{b}{\sqrt{a}} > \sqrt{a} + \sqrt{b}$.

5. $a^2 + b^2 + c^2 > ab + ac + bc$.

6. $a + \dfrac{1}{a} > 2$, whenever a does not equal 1.

7. $(a^3 + b^3)(a^5 + b^5) > (a^4 + b^4)^2$.

8. $ab(a + b) + ac(a + c) + bc(b + c) > 6abc$.

9. $ab(a + b) + ac(a + c) + bc(b + c) < 2(a^3 + b^3 + c^3)$.

10. $(a + b)(a + c)(b + c) > 8abc$.

11. $a^3 + b^3 + c^3 > 3abc$.

CHAPTER XV.

INVOLUTION AND EVOLUTION.

203. Involution is the process of raising a number to a power.

204. A number is involved by taking it as a factor as many times as there are units in the index of the required power.

205. According to Art. 70,

$$(+a)\times(+a)=+a^2,$$
$$(+a)\times(+a)\times(+a)=(+a^2)\times(+a)=+a^3,$$

and so on;

and
$$(-a)\times(-a)=+a^2,$$
$$(-a)\times(-a)\times(-a)=(+a^2)\times(-a)=-a^3,$$
$$(-a)\times(-a)\times(-a)\times(-a)=(-a^3)\times(-a)=+a^4,$$

and so on.

Hence, for the signs we have the following

Rule.

Of a positive number all the powers are positive.

Of a negative number the even powers are positive, and the odd powers negative.

INVOLUTION OF MONOMIALS.

206. To raise a monomial to any required power.

1. Find the third power of $3a^2b^3$.

$$\begin{aligned}(3a^2b^3)^3 &= 3a^2b^3\times 3a^2b^3\times 3a^2b^3 && (1)\\ &= 3\cdot 3\cdot 3\cdot a^2a^2a^2b^3b^3b^3 && (2)\\ &= 27a^6b^9 && (3)\end{aligned}$$

According to Art. 204, to raise $3a^2b^3$ to the third power, we take it as a factor three times (1); and as it makes no difference in the product in what order the factors are taken, we arrange them as in (2); performing the multiplication (§ 72) expressed in (2), we have (3). Hence,

Rule.

Multiply the exponent of each letter by the index of the required power, and prefix the required power of the numerical coefficient, remembering that the odd powers of a negative number are negative, while all other powers are positive.

NOTE 1. It follows that *the power of the product is equal to the product of the powers.*

NOTE 2. It follows from Art. 148 that a fraction is raised to a power by involving both numerator and denominator to the required power.

Find the indicated power in the following:

1. $(3a)^2$.
2. $(2xy)^3$.
3. $(a^3b^5)^4$.
4. $(5x^2y^3z^4)^3$.
5. $(-2a^7c^2)^2$.
6. $(-3b^3c^2x)^4$.
7. $(-3a^3bc^n)^3$.
8. $(2a^{4n}b^{3n}c^2)^5$.
9. $(4x^2y^2 \times 2xy)^2$.
10. $(3xyz)^n$.
11. $(-4ab^2c)^m$.
12. $\left(\frac{3c^2}{4xy}\right)^2$.
13. $\left(-\frac{2xz^2}{3b^2c^3}\right)^3$.
14. $\left(-\frac{a^2b^3}{cd^4}\right)^4$.
15. $\left(\frac{5m^2n^3}{6x^2y^3}\right)^5$.
16. $\left(-\frac{2a^3x^2}{5x^my^n}\right)^4$.
17. $\left(\frac{2a^2b^3c}{3xy^2z^4}\right)^m$.

INVOLUTION OF BINOMIALS.

207. A binomial can be raised to any power by successive multiplications. But when a high power is required, the operation is long and tedious. The **Binomial Theorem,** first developed by Sir Isaac Newton, enables us to expand a binomial to any power by a short and speedy process.

208. In order to investigate the law which governs the expansion of a binomial we will expand $a+b$ and $a-b$ to the fifth power by multiplication.

$$
\begin{array}{l}
a+b \\
a+b \\
\hline
a^2+ab \\
\qquad ab+b^2 \\
\hline
a^2+2ab+b^2 \quad \ldots\ldots\ldots\ldots \quad \text{2d power.} \\
a+b \\
\hline
a^3+2a^2b+ab^2 \\
\qquad a^2b+2ab^2+b^3 \\
\hline
a^3+3a^2b+3ab^2+b^3 \quad \ldots\ldots \quad \text{3d power.} \\
a+b \\
\hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\qquad a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
a^4+4a^3b+6a^2b^2+4ab^3+b^4 \quad \ldots\ldots \quad \text{4th power.} \\
a+b \\
\hline
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\qquad a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\
\hline
a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \quad \ldots \quad \text{5th power.}
\end{array}
$$

$$\begin{array}{llllll}
a & -\ b & & & & \\
a & -\ b & & & & \\
\hline
a^2 - & a\,b & & & & \\
\quad - & a\,b & +\ b^2 & & & \\
\hline
a^2 - 2\,a\,b & +\ b^2 & & & & \text{. . . . . . 2d power.} \\
a - b & & & & & \\
\hline
a^3 - 2\,a^2 b + & a\,b^2 & & & & \\
\quad - \ a^2 b + & 2\,a\,b^2 & -\ b^3 & & & \\
\hline
a^3 - 3\,a^2 b + & 3\,a\,b^2 & -\ b^3 & & & \text{. . . . . . 3d power.} \\
a - b & & & & & \\
\hline
a^4 - 3\,a^3 b + & 3\,a^2 b^2 - & a\,b^3 & & & \\
\quad - \ a^3 b + & 3\,a^2 b^2 - & 3\,a\,b^3 & +\ b^4 & & \\
\hline
a^4 - 4\,a^3 b + & 6\,a^2 b^2 - & 4\,a\,b^3 & +\ b^4 & & \text{. . . . . . 4th power.} \\
a - b & & & & & \\
\hline
a^5 - 4\,a^4 b + & 6\,a^3 b^2 - & 4\,a^2 b^3 + & a\,b^4 & & \\
\quad - \ a^4 b + & 4\,a^3 b^2 - & 6\,a^2 b^3 + & 4\,a\,b^4 - & b^5 & \\
\hline
a^5 - 5\,a^4 b + & 10\,a^3 b^2 - & 10\,a^2 b^3 + & 5\,a\,b^4 - & b^5 & \text{. . 5th power.}
\end{array}$$

By examining the different powers of $a + b$ and $a - b$ in these examples, we shall find the following invariable laws governing the expansion:

1st. *The leading number (that is, the first number of the binomial) begins in the first term of the power with an exponent equal to the index of the power, and its exponent decreases regularly by one in each successive term till it disappears; the following number (that is, the second number of the binomial) begins in the second term of the power with the exponent one, and its exponent increases regularly by one till in the last term it becomes the same as the index of the power.*

Thus, in the fifth power the

Exponents of a are	5,	4,	3,	2,	1.
Exponents of b are		1,	2,	3,	4, 5.

It will be noticed that the sum of the exponents of the letters in any term is equal to the index of the power.

2d. *The coefficient of the first term is one; of the second, the same as the index of the power; and universally, the coefficient of any term multiplied by the exponent of the leading quantity, and this product divided by the exponent of the following quantity increased by one, will give the coefficient of the succeeding term.*

Thus, in the fifth power, 5, the coefficient of the second term, multiplied by 4, a's exponent, and divided by 1 plus 1, b's exponent plus 1, $= \frac{5 \times 4}{2} = 10$, the coefficient of the third term.

The coefficients are repeated in the inverse order after passing the middle term or terms, so that more than half of the coefficients can be written without calculation. The number of terms is always one more than the index of the power; that is, the second power has three terms; the third power, four terms; and so on. When the number of terms is even, that is, when the index of the power is odd, the two central terms have the same coefficient.

3d. *When both terms of the binomial are positive, all the terms of the power are positive; but when the second term is negative, those terms which contain odd powers of the following quantity are negative, and all the others positive; or every alternate term, beginning with the second, is negative, and the others are positive.*

1. Expand $(x + y)^8$.

According to the law, the first term will be x^8
and the second term $+ 8x^7y$

The coefficient of the third term will be $\frac{\overset{4}{\cancel{8}} \times 7}{\cancel{2}}$
and the third term $+ 28x^6y^2$

The coefficient of the fourth term will be $\frac{28 \times \overset{2}{\cancel{6}}}{\cancel{3}}$
and the fourth term $+ 56x^5y^3$

The coefficient of the fifth term will be $\frac{\overset{14}{\cancel{56}} \times 5}{\cancel{4}}$
and the fifth term $70x^4y^4$

Having found the preceding coefficients and the coefficient of the middle term, we can write the others at once. Hence,

$$(x + y)^8 = x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$$

Expand by the binomial theorem the following:

2. $(x + y)^7$.
Ans. $x^7 + 7x^6y + 21x^5y^2 + 35x^5y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$.

3. $(a - b)^6$.

4. $(c + d)^9$.

5. $(a - b)^{10}$.

6. $(x + y)^{11}$.

7. $(a + 1)^6$.
Ans. $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1$.

NOTE. Since all the powers of 1 are 1, 1 is not written when it appears as a factor; but its exponent must be used in obtaining the coefficients.

8. $(1 - y)^8$.
Ans. $1 - 8y + 28y^2 - 56y^3 + 70y^4 - 56y^5 + 28y^6 - 8y^7 + y^8$.

9. $(c + 1)^9$.

10. $(1 - z)^{10}$.

209. When the terms of the binomial have coefficients or exponents other than 1, the theorem can be made to apply by treating each term as a single literal quantity. In the

expansion, each factor should be enclosed in a parenthesis, and after the expansion of the binomial by the binomial theorem, the work should be completed by the expansion of the enclosed factors, according to the rule for the expansion of monomials.

11. $(3a^2 - b^3)^5$.

$$(3a^2)^5 - 5(3a^2)^4(b^3) + 10(3a^2)^3(b^3)^2 - 10(3a^2)^2(b^3)^3 + 5(3a^2)(b^3)^4 - (b^3)^5$$

Expanding each factor as indicated, we have

$$243a^{10} - 405a^8b^3 + 270a^6b^6 - 90a^4b^9 + 15a^2b^{12} - b^{15}$$

12. $(2x - 5)^4$.

14. Expand $(2a + 3b)^5$.

13. Expand $(3x - 2y)^4$.

15. Expand $(2x^2 - 1)^6$.

Ans. $64x^{12} - 192x^{10} + 240x^8 - 160x^6 + 60x^4 - 12x^2 + 1$.

16. Expand $(2x - 3y)^6$.

Ans. $32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$.

17. Expand $\left(2 - \frac{x}{3}\right)^7$.

18. Expand $(a - \frac{1}{2}b)^4$. Ans. $a^4 - 2a^3b + \frac{3}{2}a^2b^2 - \frac{1}{2}ab^3 + \frac{b^4}{16}$.

19. Expand $\left(2a - \frac{b}{2}\right)^3$.

20. Expand $\left(\frac{1}{x} - \frac{1}{y}\right)^4$. Ans. $\frac{1}{x^4} - \frac{4}{x^3y} + \frac{6}{x^2y^2} - \frac{4}{xy^3} + \frac{1}{y^4}$.

21. Find the middle term of $(1 + x)^8$.

22. Find the two middle terms of $(a - b)^{13}$.

Ans. $1716a^7b^6$, and $-1716a^6b^7$.

23. Find the middle term of $\left(\frac{a}{x} + \frac{x}{a}\right)^{10}$.

24. Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

25. Find the two middle terms of $\left(3a - \frac{a^3}{6}\right)^9$.

Ans. $\frac{189a^{17}}{8}$, and $-\frac{21}{16}a^{19}$.

210. The Binomial Theorem can be applied to the expansion of a polynomial. Thus, in $a + b - c$, $a + b$ can be treated as a single term, and the quantity can be written $(a + b) - c$. In like manner, $a + b + x - y$ can be written $(a + b) + (x - y)$.

26. $(x - y - 2)^3$.

$$(x - y - 2)^3 = \{(x - y) - 2\}^3$$
$$= (x - y)^3 - 3(x - y)^2(2) + 3(x - y)(2)^2 - (2)^3$$
$$= x^3 - 3x^2y + 3xy^2 - y^3 - 6x^2 + 12xy - 6y^2 + 12x - 12y - 8$$

Note. A single letter, as a and b, might be substituted for $x - y$ and 2, and, after expanding $(a - b)^3$, the values of a and b substituted.

27. $(x - y + c)^3$.

28. $(a - 2b - c + 1)^2$.

29. $(2x - \frac{1}{2}y - m - 1)^2$.

30. $\left(\frac{a}{4} - \frac{b}{3} + 2\right)^3$.

EVOLUTION.

211. **Evolution** is the process of extracting a root of a number. It is the reverse of Involution.

212. A **Root** is one of the equal factors into which a number may be resolved (§ 12).

A root is indicated by the radical sign $\sqrt{\ }$. Thus,

$\sqrt{x}$ indicates the square root of x.
$\sqrt[3]{x}$ " " cube " "
$\sqrt[m]{x}$ " " mth " "

213. Since Evolution is the reverse of Involution, the rules for Evolution are derived at once from those of Involution. And therefore, as according to Art. 205 an odd

power of any number has the same sign as the number itself, and an even power is always positive, we have for the signs in Evolution the following

Rule.

An odd root of a number has the same sign as the number itself.

An even root of a positive number is either positive or negative.

An even root of a negative number is imaginary.

SQUARE ROOT OF ARITHMETICAL NUMBERS.

214. *To find the square root of a number is to resolve it into two equal factors, that is, to find a number which, multiplied into itself, will produce the given number.*

Numbers,	1,	10,	100,	1000.
Squares,	1,	100,	10000,	1000000.

215. Comparing the numbers above with their squares, we see that the square of any arithmetical integral number less than 10 has either one or two figures; the square of any arithmetical integral number less than 100 and over 9 has either three or four figures; and so on. That is, the square of an arithmetical number consists of twice as many figures as the root, or of one less than twice as many. Hence, to find the number of figures in the square root of an arithmetical number,

Begin at units and mark off the number into periods of two figures each, and there will be one figure in the root for each period of two figures in the square, and another figure in the root if a figure remains at the left of the full periods of the square.

216. To extract the square root of an arithmetical number.

1. Find the square root of 6889.

From the preceding explanation, it is evident that the square root of 6889 is a number of two figures, and that the tens figure of the root is the square root of the greatest perfect square in 68; that is, $\sqrt{64}$, or 8. Now, if we represent the tens of the root by a and the units by b, $a + b$ will represent the root; and the number will be

$$(a + b)^2 = a^2 + 2ab + b^2$$

Now $$a^2 = 80^2 = 6400$$

therefore $$2ab + b^2 = 6889 - 6400 = 489$$

But $$2ab + b^2 = (2a + b)b$$

If therefore 489 is divided by $2a + b$, it will give b, the units of the root. But b is unknown, and is small compared with $2a$; we can therefore use $2a = 160$ as a trial divisor. $489 \div 160$, or $48 \div 16 = 3$, a number that cannot be too small, but may be too great, because we have divided by $2a$ instead of $2a + b$. Then $b = 3$, and $2a + b = 160 + 3 = 163$, the true divisor; and $(2a + b)b = 163 \times 3 = 489$; and therefore 3 is the unit figure of the root, and 83 is the required root. The work will appear as follows:

$$\begin{array}{rl|l}
 & \overset{\frown}{68}\,\overset{\frown}{89}\,(83 & a = 80 \\
 & 64 & b = 3 \\
2a + b = 163) & 489 & \\
(2a + b)b = & 489 &
\end{array}$$

Hence, to extract the square root of an arithmetical number,

Rule.

Beginning at units, separate the number into periods of two figures each. Find the greatest square in the left-hand period, and place its root at the right. Subtract the square of this root figure from the left-hand period, and to the remainder annex the next period for a dividend. Double the

root already found for a TRIAL DIVISOR, *and, omitting the right-hand figure of the dividend, divide, and place the quotient as the next figure of the root, and also at the right of the trial divisor for the* TRUE DIVISOR. *Multiply the true divisor by this new root figure, subtract the product from the dividend, and to the remainder annex the next period, for a new dividend. Double the part of the root already found for a trial divisor, and proceed as before, until all the periods have been employed.*

NOTE 1. When a root figure is 0, annex 0 also to the trial divisor, and bring down the next period to complete the new dividend.

NOTE 2. If there is a remainder, after using all the periods in the given example, the operation may be continued at pleasure by annexing successive periods of ciphers as decimals.

2. Find the square root of 119025.

$$\begin{array}{r l}
 & \overset{\frown}{11}\,\overset{\frown}{90}\,\overset{\frown}{25}\,(345 \\
 & \underline{9} \\
64) & 290 \\
 & \underline{256} \\
685) & 3425 \\
 & \underline{3425}
\end{array}$$

We suppose at first that a represents the hundreds of the root, and b the tens; proceeding as in Ex. 1, we have 34 in the root. Then letting a represent the hundreds and tens together, that is, 34 tens, and b the units, we have $2a$, the second trial divisor, $= 64$ tens; and therefore $b = 5$; and $2a + b = 685$; and 345 is the required root.

3. Find the square root of 7527.5.

$$\begin{array}{r l}
 & \overset{\frown}{75}\,\overset{\frown}{27}.\overset{\frown}{50}\,(86.76+ \\
 & \underline{64} \\
166) & 1127 \\
 & \underline{996} \\
172.7) & 131.50 \\
 & \underline{120.89} \\
173.46) & 10.6100 \\
 & \underline{10.4076}
\end{array}$$

Find the square root of:

4. 29929.	9. 42849.	14. 1677.7216.
5. 67081.	10. 927369.	15. 360840.49.
6. 37636.	11. 290521.	16. 0.2116.
7. 762129.	12. 9703225.	17. 2330.24.
8. 401956.	13. 21418384.	18. 171819.6.

19. What is the square root of 7.5?

```
       7.50(2.73
       4
 47)   3.50
       3.29
543)   2100
       1629
546)    471
```

In this example we can only approximate to the root. By annexing successive periods of ciphers we can approximate nearer and nearer to the root. 2 is the square root to the *nearest unit;* 2.7, to the *nearest tenth;* and 2.74 (as the thousandths' figure will be more than 5), to the *nearest hundredth.*

NOTE 3. When once the decimal point has been placed in the root, no further attention need be paid to it in the remainders or in the divisors.

Find the square root of:

20. 78 to the nearest thousandth.

21. 523 to the nearest tenth.

22. 52.3 to the nearest hundredth.

NOTE 4. As a fraction is involved by involving both numerator and denominator (§ 206, Note 2), the square root of a fraction *is the square root of the numerator divided by the square root of the denominator.*

23. $\frac{9}{16}$. 24. $\frac{25}{36}$. 25. $\frac{16}{49}$. 26. $\frac{144}{169}$.

NOTE 5. If both terms of the fraction are not perfect squares, and cannot be made so, reduce the fraction to a decimal, and then find the square root of the decimal. A mixed number must be reduced to an improper fraction, or the fractional part to a decimal, before its root can be found.

27. $\frac{3}{7}$. 28. $\frac{5}{815}$. 29. $\frac{7}{11}$. 30. $8\frac{3}{7}$.

CUBE ROOT OF ARITHMETICAL NUMBERS.

217. *To find the cube root of a number is to resolve it into three equal factors; that is, to find a number which, taken three times as a factor, will produce the given number.*

Numbers,	1,	10,	100,	1000.
Cubes,	1,	1000,	1000000,	1000000000.

218. Comparing the numbers above with their cubes, we see that the cube of any arithmetical integral number less than 10 has less than three figures; the cube of any arithmetical integral number less than 100 and more than 9 has less than seven and more than three figures; and so on. That is, the *cube* of an arithmetical number consists of *three times* as many figures as the *root*, or of *one* or *two less* than three times as many. Hence, to find the number of figures in the cube root of an arithmetical number,

Begin at the right, and mark off the number into periods of three figures each, and there will be one figure in the root for each period of three figures in the cube, and if there are one or two figures besides full periods in the cube, there will be a figure in the root for this part of a period.

219. To extract the cube root of an arithmetical number.

1. Find the cube root of 79507.

From the preceding explanation, it is evident that the cube root of 79507 is a number of two figures, and that the tens figure of the root is the cube root of the greatest perfect cube in 79; that is, $\sqrt[3]{64}$, or 4. Now, if we represent the tens of the root by a and the units by b, $a + b$ will represent the root, and the number will be

$$(a + b)^3 = a^3 + 3\,a^2 b + 3\,a\,b^2 + b^3$$

Now $\qquad a^3 = 40^3 = 64000$

therefore, $\qquad 3\,a^2 b + 3\,a\,b^2 + b^3 = 79507 - 64000 = 15507$

But $\qquad 3\,a^2 b + 3\,a\,b^2 + b^3 = (3\,a^2 + 3\,a\,b + b^2)\,b$

If therefore 15507 is divided by $3a^2 + 3ab + b^2$, it will give b, the units of the root. But b, and therefore $3ab + b^2$, a part of the divisor, is unknown, and we must use $3a^2 = 4800$ as a trial divisor. $15507 \div 4800$, or $155 \div 48 = 3$, a number that cannot be too small, but may be too great, because we have divided by $3a^2$ instead of the true divisor, $3a^2 + 3ab + b^2$. Then $b = 3$, and $3a^2 + 3ab + b^2 = 4800 + 360 + 9 = 5169$, the true divisor; and $(3a^2 + 3ab + b^2)b = 5169 \times 3 = 15507$; and therefore 3 is the units figure of the root, and 43 is the required root. The work will appear as follows:

$$
\begin{array}{llrl}
 & & 79507(43 & a = 40 \\
 & & 64 & b = 3 \\
\text{Trial divisor,} & 3a^2 = 4800 & 15507 & \\
 & 3ab = 360 & & \\
 & b^2 = 9 & & \\
\text{True divisor,} & 3a^2 + 3ab + b^2 = 5169 & 15507 &
\end{array}
$$

Hence, to extract the cube root of an arithmetical number,

Rule.

Beginning at units, separate the number into periods of three figures each. Find the greatest cube in the left-hand period, and place its root at the right. Subtract this cube from the left-hand period, and to the remainder annex the next period for a dividend. Square the root figure, annex two ciphers, and multiply this result by three for a TRIAL DIVISOR; *divide the dividend by the trial divisor, and place the quotient as the next figure of the root. Multiply this root figure by the part of the root previously obtained, annex one cipher, and multiply this result by three; add the last product and the square of the last root figure to the trial divisor, and the* SUM *will be the* TRUE DIVISOR. *Multiply the true divisor by the last root figure, subtract the product from the dividend, and to the remainder annex the next period for a dividend. Find a new trial divisor, and proceed as before, until all the periods have been employed.*

Note 1. The notes under the rule in square root (§ 216) apply also to the extraction of the cube root, except that 00 must be annexed to the trial divisor when the root figure is 0.

Note 2. As the trial divisor may be much less than the true divisor, the quotient is frequently too great, and a less number must be placed in the root.

2. Find the cube root of 303464448.

$$
\begin{array}{lrr|l}
 & & & 303464448(672 \\
 & & & 216 \\
\text{1st trial divisor,} & 3a^2 = & 10800 & 87464 \\
 & 3ab = & 1260 & \\
 & b^2 = & 49 & \\
\text{1st true divisor,} & 3a^2+3ab+b^2 = & 12109 & 84763 \\
\text{2d trial divisor,} & 3a^2 = & 1346700 & 2701448 \\
 & 3ab = & 4020 & \\
 & b^2 = & 4 & \\
\text{2d true divisor,} & 3a^2+3ab+b^2 = & 1350724 & 2701448
\end{array}
$$

We suppose at first that a represents the hundreds of the root, and b the tens: proceeding as in Ex. 1, we have 67 in the root. Then, letting a represent the hundreds and tens together, that is, 67 tens, and b the units, we have $3a^2$, the 2d trial divisor, $= 1346700$; and therefore $b = 2$; and $3a^2 + 3ab + b^2$, the 2d true divisor, $= 1350724$; and 672 is the required root.

Though the 1st trial divisor is contained more than 8 times in the dividend, yet the root figure is 7.

3. Find the cube root of 129554.6.

$$
\begin{array}{r|l}
 & 129554.600(50.6+ \\
 & 125 \\
750000 & 4554.600 \\
9000 & \\
36 & \\
759036 & 4554216 \\
 & 384
\end{array}
$$

(See Note 3, § 216.)

Find the cube root of:

4. 4330747.	8. 8120601.	12. 46.78134.
5. 2924207.	9. 4789.65.	13. 5187.6423.
6. 12326391.	10. 0.07348.	14. 10073.2456.
7. 34786542.	11. 0.8754321.	15. 0.9073468.

16. What is the cube root of 7854?

```
         7854(19.8, Ans.
         1
   300 |6854
   270 |
    81 |
   ----|
   651 |5859
 108300 |995.000
   4560 |
     64 |
 -------|
 112924 |903.392
        |-------
          91.608
```

In this example we can only approximate to the root. By annexing successive periods of ciphers we can approximate nearer and nearer to the root. 20 is the cube root to the *nearest unit;* 19.8, to the *nearest tenth.*

Find the cube root of:

17. 10 to the nearest hundredth.

18. 560 to the nearest tenth.

19. 0.08 to the nearest thousandth.

NOTE 3. As a fraction is involved by involving both numerator and denominator (§ 206, Note 2), the cube root of a fraction is *the cube root of the numerator divided by the cube root of the denominator.*

20. $\frac{27}{64}$. 21. $\frac{125}{216}$. 22. $\frac{343}{729}$. 23. $\frac{1536}{2187}$.

NOTE 4. If both terms of the fraction are not perfect cubes, and cannot be made so, reduce the fraction to a decimal, and then find the cube root of the decimal. A mixed number must be reduced to an improper fraction, or the fractional part to a decimal, before its root can be found.

24. $\frac{5}{7}$. 25. $\frac{6}{11}$. 26. $5\frac{2}{3}$. 27. $17\frac{3}{8}$.

EVOLUTION OF MONOMIALS.

220. As **Evolution** is the reverse of Involution, and since to involve a monomial (§ 206) we multiply the exponent of each letter by the index of the required power, and prefix the required power of the numerical coefficient, therefore, to find the root of a monomial,

Rule.

Divide the exponent of each letter by the index of the required root, and prefix the required root of the numerical coefficient.

NOTE 1. The rule for the signs is given in Art. 213. As an even root of a positive number may be either positive or negative, we prefix to such a root the sign $\pm$; read, plus or minus.

NOTE 2. It follows from this rule that *the root of the product of several factors is equal to the product of the roots.* Thus, $\sqrt{36} = \sqrt{4}\ \sqrt{9} = 6$.

Perform the operation indicated in the following examples:

1. $\sqrt[3]{27\,x^3\,y^6}$.
2. $\sqrt{9\,a^2\,b^4}$.
3. $\sqrt[4]{16\,m^4\,n^8\,p^{12}}$.
4. $\sqrt[3]{-8\,a^3\,b^9}$.
5. $\sqrt[5]{32\,x^{10}\,y^5}$.
6. $\sqrt[4]{81\,a^4\,b^8}$.
7. $\sqrt[5]{-243\,a^{10}\,b^5}$.
8. $\sqrt[3]{125\,x^6\,y^3}$.
9. $\sqrt[4]{625\,a^4\,b^8\,c^{12}}$.
10. $\sqrt[3]{-729\,x^9\,y^3}$.

NOTE 3. As a fraction is involved by involving both numerator and denominator (§ 206, Note 2), a fraction must be evolved by evolving both numerator and denominator.

11. $\sqrt{\dfrac{9\,x^2}{16\,y^4}}$. Ans. $\pm\dfrac{3\,x}{4\,y^2}$.

12. $\sqrt[3]{-\dfrac{8\,a^3\,b^6}{27\,x^3}}$.

13. $\sqrt{\dfrac{16\,x^2\,y^4}{25\,a^4\,b^6}}$.

14. $\sqrt[7]{\frac{128}{a^{63}b^{56}}}$.

15. $\sqrt[10]{\frac{a^{30}x^{50}}{b^{100}}}$.

16. $\sqrt[9]{\frac{a^{18}}{b^{27}c^{88}}}$.

17. $\sqrt[8]{\frac{a^{24}x^{8}}{b^{16}y^{40}}}$.

SQUARE ROOT OF POLYNOMIALS.

221. In order to discover a method for extracting the square root of a polynomial, we will consider the relation of $a + b$ to its square, $a^2 + 2ab + b^2$. The first term of the square contains the square of the first term of the root; therefore the square root of the first term of the square will be the first term of the root. The second term of the square contains twice the product of the two terms of the root; therefore, if the second term of the square, $2ab$, is divided by twice the first term of the root, $2a$, we shall have the second term of the root b. Now, $2ab + b^2 = (2a + b)b$; therefore, if to the trial divisor $2a$ we add b, when it has been found, and then multiply the corrected divisor by b, the product will be equal to the remaining terms of the power after a^2 has been subtracted.

The process will appear as follows:

$$\begin{array}{r|l} & a^2 + 2ab + b^2\,(a + b \\ & \underline{a^2\qquad\qquad\quad} \\ 2a + b) & 2ab + b^2 \\ & \underline{2ab + b^2} \end{array}$$

Having written a, the square root of a^2, in the root, we subtract its square (a^2) from the given polynomial, and have $2ab + b^2$ left. Dividing the first term of this remainder, $2ab$, by $2a$, which is double the term of the root already found, we obtain b, the second term of the root, which we add both to the root and to the divisor. If the product of this corrected divisor and the last term of the root is subtracted from $2ab + b^2$, nothing remains.

222. Since a polynomial can always be written and involved like a binomial, as shown in Art. 210, we can apply the process explained in the preceding article to finding the root, when this root consists of any number of terms.

1. Find the square root of $a^2 + 2ab + b^2 - 2ac - 2bc + c^2$.

$$\begin{array}{l}
a^2 + 2ab + b^2 - 2ac - 2bc + c^2 \,(\, a + b - c \\
a^2 \\
\hline
2a + b)\, 2ab + b^2 \\
\qquad\quad 2ab + b^2 \\
\hline
2a + 2b - c) - 2ac - 2bc + c^2 \\
\qquad\qquad\quad - 2ac - 2bc + c^2 \\
\hline
\end{array}$$

Proceeding as before, we find the first two terms of the root $a + b$. Considering $a + b$ as a single number, we divide the remainder $-2ac - 2bc + c^2$ by twice this root, and obtain $-c$, which we write both in the root and in the divisor. If this corrected divisor is multiplied by $-c$, and the product subtracted from the dividend, nothing remains.

Hence, to extract the square root of a polynomial,

Rule.

Arrange the terms according to the powers of some letter.

Find the square root of the first term, and write it as the first term of the root, and subtract its square from the given polynomial.

Divide the remainder by double the root already found, and annex the result both to the root and to the divisor.

Multiply the corrected divisor by this last term of the root, and subtract the product from the last remainder. Proceed as before with the remainder, if there is any.

Find the square root of:

2. $4x^2 + 8xy + 4y^2 + 4xz + 4yz + z^2$.

3. $a^4 - 2a^3 + 3a^2 - 2a + 1$.

4. $4a^4 + 4a^3 - 7a^2 - 4a + 4$. Ans. $2a^2 + a - 2$.

5. $4x^2 + 9y^2 + 25z^2 + 12xy - 30yz - 20xz$.

6. $16x^6 + 16x^7 - 4x^8 - 4x^9 + x^{10}$.

Ans. $4x^3 + 2x^4 - x^5$.

7. $x^6 - 22x^4 + 34x^3 + 121x^2 - 374x + 289$.

8. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4$.

9. $4x^4 + 4x^2y^2 - 12x^2z^2 + y^4 - 6y^2z^2 + 9z^4$.

10. $6ab^2c - 4a^2bc + a^2b^2 + 4a^2c^2 + 9b^2c^2 - 12abc^2$.

11. $-6b^2c^2 + 9c^4 + b^4 - 12c^2a^2 + 4a^4 + 4a^2b^2$.

Ans. $2a^2 + b^2 - 3c^2$.

12. $4x^4 + 9y^4 + 13x^2y^2 - 6xy^3 - 4x^3y$.

13. $67x^2 + 49 + 9x^4 - 70x - 30x^3$. Ans. $3x^2 - 5x + 7$.

14. $1 - 4x + 10x^2 - 20x^3 + 25x^4 - 24x^5 + 16x^6$.

15. $6acx^5 + 4b^2x^4 + a^2x^{10} + 9c^2 - 12bcx^2 - 4abx^7$.

Ans. $ax^5 - 2bx^2 + 3c$.

16. $\frac{16y^2}{x^2} - \frac{32y}{x} + 24 - \frac{8x}{y} + \frac{x^2}{y^2}$.

$$\frac{16y^2}{x^2} - \frac{32y}{x} + 24 - \frac{8x}{y} + \frac{x^2}{y^2}\left(\frac{4y}{x} - 4 + \frac{x}{y}\right.$$

$$\frac{16y^2}{x^2}$$

$$\left.\frac{8y}{x} - 4\right) - \frac{32y}{x} + 24$$

$$-\frac{32y}{x} + 16$$

$$\left.\frac{8y}{x} - 8 + \frac{x}{y}\right) 8 - \frac{8x}{y} + \frac{x^2}{y^2}$$

$$8 - \frac{8x}{y} + \frac{x^2}{y^2}$$

17. $\frac{x^2}{y^2} + \frac{10x}{y} + 25$.

18. $\dfrac{a^4}{64} + \dfrac{a^3}{8} - a + 1.$

19. $x^4 - 6x^3 + \frac{29}{3}x^2 - 2x + \frac{1}{9}.$

20. $\dfrac{9a^2}{x^2} - \dfrac{6a}{5x} + \dfrac{101}{25} - \dfrac{4x}{15a} + \dfrac{4x^2}{9a^2}.$

NOTE 1. According to the principles of Art. 200, the signs of the answers given above may all be changed, and still be correct.

NOTE 2. *No binomial can be a perfect square.* For the square of a monomial is a monomial, and the square of the polynomial with the least number of terms, that is, of a binomial, is a trinomial.

NOTE 3. By the rule for extracting the square root, any root whose index is any power of 2 can be obtained by successive extractions of the square root. Thus, the fourth root is the square root of the square root; the eighth root is the square root of the square root of the square root; and so on.

Find the fourth root of:

21. $16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8.$ Ans. $2x - y^2.$

22. $a^8 - 12a^6b + 54a^4b^2 - 108a^2b^3 + 81b^4.$

23. $\dfrac{1}{16} + \dfrac{x}{2} + \dfrac{3x^2}{2} + 2x^3 + x^4.$

24. $\dfrac{1}{x^4} + \dfrac{4}{x^3y} + \dfrac{6}{x^2y^2} + \dfrac{4}{xy^3} + \dfrac{1}{y^4}.$

223. To find any root of a polynomial.

Since, according to the Binomial Theorem, when the terms of a power are arranged according to the power of some letter beginning with its highest power, the first term contains the first term of the root raised to the given power, therefore, if we take the required root of the first term, we shall have the first term of the root. And since the second term of the power contains the second term of the root multiplied by the next inferior power of the first term of the root with a coefficient equal to the index of the root, therefore, if we divide the second term of the power by the first term of the root raised to the next inferior power with a coefficient equal to the index of the root, we shall have the second term of the root. In accordance with these principles, to find any root of a polynomial we have the following

Rule.

Arrange the terms according to the powers of some letter.

Find the required root of the first term, and write it as the first term of the root.

Divide the second term of the polynomial by the first term of the root raised to the next inferior power and multiplied by the index of the root.

Involve the whole of the root thus found to the given power, and subtract it from the polynomial.

If there is any remainder, divide its first term by the divisor first found, and the quotient will be the third term of the root.

Proceed in this manner till the power obtained by involving the root is equal to the given polynomial.

NOTE 1. This rule verifies itself. For the root, whenever a new term is added to it, is involved to the given power; and whenever the root thus involved is equal to the given polynomial, it is evident that the required root is found.

NOTE 2. As *powers* and *roots* are correlative words, we have used the phrase *given power*, meaning the power whose index is equal to the index of the required root, and the phrase *next inferior power*, meaning that power whose index is one less than the index of the required root.

1. Find the cube root of $a^6 - 3a^5 + 5a^3 - 3a - 1$.

$$
\begin{array}{lrl}
\text{Constant divisor,} & 3a^4\,) & a^6 - 3a^5 + 5a^3 - 3a - 1\ (a^2 - a - 1 \\
 & & \underline{a^6 - 3a^5 + 3a^4 - a^3} \\
 & & \quad -3a^4, \quad \text{1st term of remainder.} \\
 & & \underline{a^6 - 3a^5 + 5a^3 - 3a - 1}
\end{array}
$$

The first term of the root is a^2, the cube root of a^6. a^2 raised to the next inferior power, that is, to the second power, with the coefficient 3, the index of the root, gives $3a^4$, which is the constant divisor. $-3a^5$, the second term of the polynomial, divided by $3a^4$, gives $-a$, the second term of the root. $(a^2 - a)^3 = a^6 - 3a^5 + 3a^4 - a^3$; and subtracting this from the polynomial, we have $-3a^4$ as the first term of the remainder. $-3a^4$ divided by $3a^4$ gives -1, the third term of the root. $(a^2 - a - 1)^3 =$ the given polynomial, and therefore the correct root has been found.

2. Find the fourth root of $81\,a^8 - 108\,a^6x^3 + 54\,a^4x^6 - 12\,a^2x^9 + x^{12}$.

$$4\,(3\,a^2)^3 = 108\,a^6)\;81\,a^8 - 108\,a^6x^3 + 54\,a^4x^6 - 12\,a^2x^9 + x^{12}\;(3\,a^2 - x^3$$
$$\underline{81\,a^8 - 108\,a^6x^3 + 54\,a^4x^6 - 12\,a^2x^9 + x^{12}}$$

3. Find the cube root of $x^3 + 3\,x^2y + 3\,xy^2 + y^3 - 3\,x^2z - 6\,xyz - 3\,y^2z + 3\,xz^2 + 3\,yz^2 - z^3$.

4. Find the fourth root of $16\,b^8 - 96\,b^6c + 216\,b^4c^2 - 216\,b^2c^3 + 81\,c^4$.

Find the cube root of:

5. $a^3x^3 - 3\,a^2x^2y^2 + 3\,axy^4 - y^6$. Ans. $ax - y^2$.

6. $1 + 3\,x + 6\,x^2 + 7\,x^3 + 6\,x^4 + 3\,x^5 + x^6$.

7. $1 - 6\,x + 21\,x^2 - 44\,x^3 + 63\,x^4 - 54\,x^5 + 27\,x^6$.

8. $a^3 + 6\,a^2b - 3\,a^2c + 12\,ab^2 - 12\,abc + 3\,ac^2 + 8\,b^3 - 12\,b^2c + 6\,bc^2 - c^3$. Ans. $a + 2\,b - c$.

9. $8\,x^6 + 12\,x^5 - 30\,x^4 - 35\,x^3 + 45\,x^2 + 27\,x - 27$.
Ans. $2\,x^2 + x - 3$.

10. $\dfrac{x^3}{8} - \dfrac{3\,x^2}{4} + \dfrac{3\,x}{2} - 1$. Ans. $\dfrac{x}{2} - 1$.

11. $8\,x^3 - 4\,x^2y^2 + \frac{2}{3}\,xy^4 - \dfrac{y^6}{27}$.

12. $\dfrac{x^3}{y^3} + \dfrac{6\,x^2}{y^2} + \dfrac{9\,x}{y} - 4 - \dfrac{9\,y}{x} + \dfrac{6\,y^2}{x^2} - \dfrac{y^3}{x^3}$.
Ans. $\dfrac{x}{y} + 2 - \dfrac{y}{x}$.

13. $\dfrac{x^3}{27} - \dfrac{x^2}{3} + 2\,x - 7 + \dfrac{18}{x} - \dfrac{27}{x^2} + \dfrac{27}{x^3}$.

14. $\dfrac{x^3}{a^3} - \dfrac{12\,x^2}{a^2} + \dfrac{54\,x}{a} - 112 + \dfrac{108\,a}{x} - \dfrac{48\,a^2}{x^2} + \dfrac{8\,a^3}{x^3}$.

15. $\dfrac{64\,a^3}{x^3} - \dfrac{192\,a^2}{x^2} + \dfrac{240\,a}{x} - 160 + \dfrac{60\,x}{a} - \dfrac{12\,x^2}{a^2} + \dfrac{x^3}{a^3}$.
Ans. $\dfrac{4\,a}{x} - 4 + \dfrac{x}{a}$.

CHAPTER XVI.

THEORY OF INDICES.

224. In Art. 72 we have proved that $a^3 \times a^2 = a^5$, and inferred that the same principle is true whatever the indices may be. We now propose to prove that, if m and n are positive integers,

$$\text{I.} \quad a^m \times a^n = a^{m+n}.$$
$$\text{II.} \quad a^m \div a^n = a^{m-n}.$$
$$\text{III.} \quad (a^m)^n = a^{mn}.$$
$$\text{IV.} \quad (ab)^m = a^m b^m.$$

225. *To prove* (I.), $a^m \times a^n = a^{m+n}$.

By Art. 11, $a^m = a \times a \times a \ldots$ to m factors,
and $a^n = a \times a \times a \ldots$ to n factors.
Therefore,

$$\begin{aligned} a^m \times a^n &= (a \times a \times a \ldots \text{to } m \text{ factors})(a \times a \times a \ldots \text{to } n \text{ factors}) \\ &= a \times a \times a \ldots \text{to } m + n \text{ factors} \\ &= a^{m+n} \end{aligned}$$

If p is a positive integer, then

$$a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p},$$

and so for any number of factors.

226. *To prove* (II.), $a^m \div a^n = a^{m-n}$, *when* $m > n$.

$$\begin{aligned} a^m \div a^n = \frac{a^m}{a^n} &= \frac{a \times a \times a \ldots \text{to } m \text{ factors}}{a \times a \times a \ldots \text{to } n \text{ factors}} \\ &= a \times a \times a \ldots \text{to } m - n \text{ factors} \\ &= a^{m-n} \end{aligned}$$

227. *To prove* (III.), $(a^m)^n = a^{mn}$.

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \ldots \text{to } n \text{ factors}\\ &= a^{m+m+m} \ldots \text{to } n \text{ terms}\\ &= a^{mn}\end{aligned}$$

228. *To prove* (IV.), $(ab)^m = a^m b^m$.

$$\begin{aligned}(ab)^m &= (ab \times ab \times \ldots \text{to } m \text{ factors}\\ &= (a \times a \times \ldots \text{to } m \text{ factors})\ (b \times b \times \ldots \text{to } m \text{ factors})\\ &= a^m b^m\end{aligned}$$

229. It now remains to show that these laws (§§ 224–228) hold for fractional and negative indices as well, and hence are universally true. We assume that $a^{\frac{m}{n}}$, a^0, a^{-m} conform to the fundamental law, $a^m \times a^n = a^{m+n}$, and accept the meaning to which this assumption leads.

For example, to find the meaning of $a^{\frac{1}{3}}$. According to our assumption, we must have

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a.$$

Hence, $a^{\frac{1}{3}}$ must be such that its third power is a, that is, $a^{\frac{1}{3}} = \sqrt[3]{a}$.

230. *To find the meaning of* $a^{\frac{m}{n}}$, *when* m *and* n *are positive integers.*

By Art. 229, $\quad a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \ldots$ to n factors

$$\begin{aligned}&= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}\ldots \text{to } n \text{ terms}}\\ &= a^{\frac{m+m+m\ldots \text{to } n \text{ terms}}{n}}\\ &= a^{\frac{mn}{n}} = a^m\end{aligned}$$

Hence, $a^{\frac{m}{n}}$ must be such that its nth power is a^m, that is, $a^{\frac{m}{n}} = \sqrt[n]{a^m}$. In particular, if $m = 1$,

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Hence, $a^{\frac{m}{n}}$ means the nth root of the mth power of a; that is, in a fractional index the numerator denotes a power, and the denominator a root.

Thus, $$a^{\frac{3}{4}} = \sqrt[4]{a^3}, \quad a^{\frac{2}{3}} = \sqrt[3]{a^2}.$$

231. *To find the meaning of* a^0.

By Art. 229, $$a^0 \times a^m = a^{0+m} = a^m$$

$$\therefore\ a^0 = \frac{a^m}{a^m} = 1$$

Hence, any number with zero for an exponent is equivalent to 1.

232. *To find the meaning of* a^{-m}, *where* m *has any positive value.*

By Art. 229, $$a^{-m} \times a^m = a^{-m+m} = a^0 = 1$$

Hence, $$a^{-m} = \frac{1}{a^m}, \quad \text{and} \quad a^m = \frac{1}{a^{-m}}$$

233. It follows that *a factor may be transferred from the numerator of a fraction to its denominator, or vice versa, provided the sign of the exponent of the factor is changed from + to −, or − to +.*

Express with positive indices:

1. x^{-3}.
2. $3a^{-2}b^{-3}$.
3. $4x^{-2}a^3$.
4. $\dfrac{1}{4a^{-2}}$.
5. $\dfrac{1}{5x^{-\frac{1}{2}}}$.
6. $\dfrac{3a^{-3}x^2}{5y^2c^{-4}}$.
7. $\dfrac{a^5b^{-1}}{x^{-3}y^2}$.
8. $\dfrac{5x^{-1}y^2z^{-3}}{6a^{-2}bc^{-3}}$.
9. $\dfrac{a^{-2}b^{-2}}{c^{-2}d^{-2}}$.
10. $\dfrac{a^{-2}b^{-3}c^{-1}}{x^2y^{-2}z^{-1}}$.
11. $\dfrac{x^2y^{-1}z}{a^{-1}bc^{-4}}$.
12. $\dfrac{b(x-y)^{-1}}{x+y}$.
13. $\dfrac{ax^{-3}z^2}{b^{-1}c^{-2}y}$.
14. $\dfrac{(a+b)(a-b)^{-2}}{(a-b)^{-1}(a+b)^{-2}}$.
15. $\dfrac{a^{-m}b^n}{x^{-m}y^nz^{-2}}$.
16. $\dfrac{5^{-1}a^{-2}b^3x^{-2}}{8^{-1}m^{-2}ny^{-3}}$.

Express with radical signs and positive indices:

17. $x^{\frac{3}{5}}$.

18. $\dfrac{2}{b^{-\frac{3}{4}}}$.

19. $a^{-\frac{1}{2}}$.

20. $\dfrac{c^{-\frac{1}{3}}}{2}$.

21. $2\,a^{-\frac{1}{x}}$.

22. $\dfrac{1}{x^{-\frac{1}{x}}}$.

Find the value of:

23. $16^{\frac{3}{4}}$.

24. $4^{-\frac{5}{2}}$.

25. $8^{-\frac{2}{3}}$.

26. $(\frac{8}{27})^{-\frac{1}{3}}$.

27. $(\frac{81}{16})^{\frac{3}{4}}$.

28. $(\frac{32}{343})^{-\frac{7}{5}}$.

234. *To prove* $a^m \div a^n = a^{m-n}$ *for all values of* m *and* n.

By Art. 232, $$a^m \div a^n = \frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}$$

Thus, $$a^2 \div a^5 = a^{2-5} = a^{-3} = \frac{1}{a^3}$$

$$a^3 \div a^{-\frac{3}{2}} = a^{3+\frac{3}{2}} = a^{\frac{9}{2}}$$

235. According to the principles established, reduce the following expressions to their simplest form:

1. $a^9 \times a^{-2}$.
2. $a^{-5} \times a^{-2}$.
3. $a^6 \times a^{-6}$.
4. $a^{-7} \times a^4$.
5. $a^3 x^4 y^2 \times a^{-2} x^{-4} y^2$.
6. $a^5 x^3 y^{-3} \times a^{-5} x^{-3} y^3$.
7. $3x^{-1} y^{-2} z \times 2 x y^2 z^{-1}$.
8. $15 a^3 b^2 x \times 4 a^{-2} b x^2 y$.
9. $7 a^{-\frac{1}{2}} \times 3 a^{-1}$.
10. $\sqrt[2a]{x} \times \sqrt[a]{x^2}$.
11. $x^7 \div x^8$.
12. $x^5 \div x^{-2}$.
13. $x^{-5} \div x^3$.
14. $x^{-3} \div x^{-2}$.
15. $a^5 \div a^{-5}$.
16. $a^{-5} \div a^5$.
17. $16 b^2 c \div 4 b^3 c^2$.
18. $\sqrt[5]{a^{-x}} \div \sqrt[5]{a^{-2x}}$.
19. $\sqrt[a]{x} \div \sqrt[2a]{x^3}$.
20. $\sqrt[4]{a^n} \times \sqrt[3]{a^n} \div \sqrt[12]{a^{5n}}$.
21. $\dfrac{\sqrt{x^3} \times \sqrt[3]{y^2}}{\sqrt[6]{y^{-2}} \times \sqrt[4]{x^6}}$.
22. $\dfrac{2 a^{\frac{1}{2}} \times a^{\frac{2}{3}} \times 6 a^{-\frac{7}{3}}}{9 a^{\frac{5}{3}} \times a^{\frac{3}{2}}}$.

Ans. $\dfrac{4}{3a}$.

236. *To prove that $(a^m)^n = a^{mn}$ is true for all values of m and n.*

This has already been shown to be true when m and n are positive integers (§ 227).

I. *Let n be a positive fraction, $\frac{p}{q}$, where p and q are positive integers.* Now, whatever be the value of m,

By Art. 230, $\quad (a^m)^n = (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p}$

By Art. 227, $\quad = \sqrt[q]{a^{mp}}$

By Art. 230, $\quad = a^{\frac{mp}{q}}$

II. *Let n be negative, and equal to $-p$, where p is positive and m unrestricted, as before.* Then,

By Art. 232, $\quad (a^m)^n = (a^m)^{-p} = \dfrac{1}{(a^m)^p}$

By Art. 227, $\quad = \dfrac{1}{a^{mp}}$

By Art. 233, $\quad = a^{-mp}$

Hence, $\quad (a^m)^n = a^{mn}$ for *all values* of m and n.

It follows that $\quad (a^m)^{\frac{1}{n}} = (a^{\frac{1}{n}})^m = (a)^{\frac{m}{n}}$

That is, the nth root of the mth power of a is eqnivalent to the mth power of the nth root of a.

Also, $\quad (a^{\frac{1}{m}})^{\frac{1}{n}} = (a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}}$

That is, the nth root of the mth root of a, or the mth root of the nth root of a, is equal to the mnth root of a.

Thus, $\quad (b^{\frac{1}{3}})^{\frac{5}{7}} = b^{\frac{1}{3} \times \frac{5}{7}} = b^{\frac{5}{21}}$

$$[(a^{-2})^3]^5 = (a^{-6})^5 = a^{-30}$$

237. *To prove that $(ab)^n = a^n b^n$, whatever be the value of n.*

We have already proved this to be true where n is a positive integer (§ 228).

Let n be a positive fraction, $\frac{p}{q}$, where p and q are positive integers.

Then $$(ab)^n = (ab)^{\frac{p}{q}}$$

Now by Art 236, $$[(ab)^{\frac{p}{q}}]^q = (ab)^p$$

By Art. 228, $$= a^p b^p$$

$$= (a^{\frac{p}{q}} b^{\frac{p}{q}})^q$$

$$\therefore \ (ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}$$

Let n have any negative value, say $-m$, where m is a positive integer. Then,

$$(ab)^n = (ab)^{-m} = \frac{1}{(ab)^m}$$

$$= \frac{1}{a^m b^m} = a^{-m} b^{-m}$$

238. It follows that

$$(abc)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}} c^{\frac{1}{n}}$$

$$\therefore \ \sqrt[n]{abc} = \sqrt[n]{a} \cdot \sqrt[n]{b} \cdot \sqrt[n]{c}$$

That is, the nth root of the product of several numbers is equal to the product of nth roots of those numbers.

239. It should be observed that, in the proof above, the numbers a and b are *wholly unrestricted*, and may themselves have indices.

Express in the simplest form, without negative indices, the following examples:

1. $(x^{\frac{1}{2}} y^{-\frac{1}{2}}) \div (x^2 y^{-1})^{-\frac{1}{3}}$.

$$(x^{\frac{1}{2}} y^{-\frac{1}{2}})^{\frac{4}{3}} \div (x^2 y^{-1})^{-\frac{1}{3}} = x^{\frac{2}{3}} y^{-\frac{2}{3}} \div x^{-\frac{2}{3}} y^{\frac{1}{3}} = x^{\frac{4}{3}} y^{-1}$$

2. $\left(\dfrac{a^{\frac{2}{3}} \sqrt{b^{-1}}}{b \sqrt[3]{a^{-2}}} \div \sqrt{\dfrac{a \sqrt{b^{-4}}}{b \sqrt{a^{-2}}}}\right)^6$.

$$\left(\frac{a^{\frac{2}{3}} \sqrt{b^{-1}}}{b \sqrt[3]{a^{-2}}} \div \sqrt{\frac{a \sqrt{b^{-4}}}{b \sqrt{a^{-2}}}}\right)^6 = \left(\frac{a^{\frac{2}{3}} b^{-\frac{1}{2}}}{b a^{-\frac{2}{3}}} \div \frac{a^{\frac{1}{2}} b^{-1}}{b^{\frac{1}{2}} a^{-\frac{1}{2}}}\right)^6$$

$$= \left(\frac{a^{\frac{2}{3}} b^{-\frac{1}{2}}}{b a^{-\frac{2}{3}}} \times \frac{b^{\frac{1}{2}} a^{-\frac{1}{2}}}{a^{\frac{1}{2}} b^{-1}}\right)^6$$

$$= (a^{\frac{2}{3}-\frac{1}{2}+\frac{2}{3}-\frac{1}{2}} b^{-\frac{1}{2}+\frac{1}{2}-1+1})^6$$

$$= (a^{\frac{1}{3}})^6 = a^2$$

3. $(x^a y^{-b})^3 \times (x^3 y^2)^{-a}$.

4. $\left(\dfrac{16\,x^2}{y^{-2}}\right)^{-\frac{1}{4}}$.

5. $\left(\dfrac{27\,x^3}{8\,a^{-3}}\right)^{-\frac{2}{3}}$. Ans. $\dfrac{4}{9\,a^2x^2}$.

6. $\left(\dfrac{a^{-\frac{1}{2}}}{4\,c^2}\right)^{-2}$.

7. $\left\{\sqrt[4]{(x^{-\frac{2}{3}}\,y^{\frac{1}{2}})^3}\right\}^{-\frac{2}{3}}$. Ans. $\dfrac{x^{\frac{1}{3}}}{y^{\frac{1}{4}}}$.

8. $\sqrt[4]{x\,\sqrt[3]{x^{-1}}}$.

9. $(4\,a^{-2} \div 9\,x^2)^{-\frac{1}{2}}$.

10. $(x \div \sqrt[n]{x})^n$.

11. $(x \times \sqrt[n]{x^{-\frac{1}{n}}})^{\frac{n^2}{1-n}}$. Ans. $\dfrac{1}{x^{n+1}}$.

12. $(\sqrt[b]{x^b} \div \sqrt[a]{x})^{\frac{1}{1-a}}$.

13. $\sqrt{a^{-2}\,b} \times \sqrt[3]{a\,b^{-3}}$.

14. $\sqrt[3]{a\,b^{-1}c^{-2}} \times (a^{-1}b^{-2}\,c^{-4})^{-\frac{1}{6}}$. Ans. $a^{\frac{1}{2}}$.

15. $\sqrt[6]{a^{4b}\,x^6} \times (a^{\frac{2}{3}}\,x^{-1})^{-b}$.

16. $\sqrt[3]{x^{-1}\,\sqrt{y^3}} \div \sqrt{y\,\sqrt[5]{x}}$.

17. $(a^{-\frac{1}{2}}\,\sqrt[3]{x})^{-3} \times \sqrt{x^{-2}\,\sqrt{a^{-6}}}$.

18. $\sqrt[n]{a^{n+k}\,b^{2n-k}} \div (a^{\frac{1}{n}}\,b^{-\frac{1}{n}})^k$. Ans. $a\,b^2$.

19. $\sqrt[3]{(a+b)^5} \times (a+b)^{-\frac{2}{3}}$.

20. $\{(x-y)^{-3}\}^n \div \{(x+y)^n\}^3$.

21. $\left(\dfrac{a^{-2}b}{a^3\,b^{-4}}\right)^{-3} \div \left(\dfrac{a\,b^{-1}}{a^{-3}\,b^2}\right)^5$. Ans. $\dfrac{1}{a^5}$.

22. $\left\{\dfrac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}} \cdot \left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right\}^6$. Ans. $b^{\frac{3}{2}}$.

23. $\left(a^{-\frac{1}{2}}\,x^{\frac{1}{3}}\sqrt{a\,x^{\frac{1}{3}}\,\sqrt[4]{x^{\frac{4}{3}}}}\right)^{\frac{1}{3}}$.

24. $\sqrt[4]{(a+b)^6} \times (a^2-b^2)^{-\frac{1}{2}}$.

25. $\left(\dfrac{a^{-3}}{b^{-\frac{2}{3}}\,c}\right)^{-\frac{2}{3}} \div \left(\dfrac{\sqrt{a^{-\frac{1}{2}} \cdot \sqrt[6]{b^3}}}{a^2\,c^{-1}}\right)^{-2}$.

26. $\left(\dfrac{a^{-\frac{2}{3}}\,x^{\frac{1}{2}}}{x^{-1}\,a}\right)^2 \div \sqrt[3]{\dfrac{a^{-1}}{x^{-3}}}$. Ans. $\dfrac{x^2}{a^3}$.

27. $\left(\sqrt[5]{\dfrac{a^{\frac{1}{2}}x^{-2}}{x^{\frac{1}{2}}a^{-2}}} \times \sqrt[3]{\dfrac{a\sqrt{x}}{x^{-1}\sqrt{a}}}\right)^{-4}.$ Ans. $\dfrac{1}{a^{\frac{8}{3}}}.$

28. $\dfrac{\sqrt[3]{(a^3b^3+a^6)}}{\sqrt[3]{(b^3-a^3b^3)^{-1}}}.$ 29. $(a^{n^2-1})^{\frac{n}{n+1}} + \dfrac{\sqrt[n]{a^{2n}}}{a}.$

30. $\left(x^{\frac{n}{n+1}}\right)^{n^2-1} + \dfrac{\sqrt{x^{2n}}}{x}.$

31. $\left\{\dfrac{a^{p-q}}{\sqrt[q]{a^{q^2-pq}}} \times a^{2(p-q)}\right\}^n.$ Ans. $a^{4n(p-q)}.$

32. $\left(x^{\frac{a}{b}}y^{-1}\right)^b \div \left(\dfrac{x^{a^2-b^2}}{y^{ab+b^2}}\right)^{\frac{1}{a+b}}.$

33. $\left(\dfrac{x^{-2}y^3}{x^3y^{-2}}\right)^{\frac{1}{5}} \times \left(\dfrac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1}.$

34. $\left(\dfrac{y^{-3}}{x^{\frac{2}{7}}z^{-1}}\right)^{-\frac{3}{2}} \times \left(\dfrac{y^{\frac{14}{3}}x^{-1}}{z^{-\frac{21}{4}}}\right)^{\frac{2}{7}}.$ Ans. $x^{\frac{1}{7}}y^{\frac{35}{6}}.$

35. $\dfrac{2^n \times (2^{n-1})^n}{2^{n+1} \times 2^{n-1}} \times \dfrac{1}{4^{-n}}.$

36. $\dfrac{2^{n+1}}{(2^n)^{n-1}} \div \dfrac{4^{n+1}}{(2^{n-1})^{n+1}}.$ Ans. $\frac{1}{4}.$

240. Since the index laws are universally true, all the operations of multiplication, division, involution, and evolution are applicable to expressions which contain fractional and negative indices.

In working examples, orderly arrangement must be observed. The descending powers of x are

$$\ldots\ x^5,\ x^4,\ x^3,\ x^2,\ x,\ \frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ \frac{1}{x^4},\ \frac{1}{x^5},\ \ldots$$

or (§ 232),

$$\ldots\ x^5,\ x^4,\ x^3,\ x^2,\ x,\ x^{-1},\ x^{-2},\ x^{-3},\ x^{-4},\ x^{-5},\ \ldots$$

EXAMPLES.

1. Multiply $3x^{-\frac{1}{3}} + x + 2x^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - 2$.

Arrange in descending powers of x:

$$\begin{array}{llll}
x & + 2x^{\frac{2}{3}} + 3x^{-\frac{1}{3}} & & \\
\underline{x^{\frac{1}{3}} - 2} & & & \\
x^{\frac{4}{3}} + 2x + 3 & & & \\
\quad - 2x & - 4x^{\frac{2}{3}} & - 6x^{-\frac{1}{3}} & \\
\hline
x^{\frac{4}{3}} & - 4x^{\frac{2}{3}} + 3 & - 6x^{-\frac{1}{3}} &
\end{array}$$

2. Divide $16a^{-3} - 6a^{-2} + 5a^{-1} + 6$ by $1 + 2a^{-1}$.

$$\begin{array}{rl}
2a^{-1} + 1\,) & 16a^{-3} - 6a^{-2} + 5a^{-1} + 6\,(\,8a^{-2} - 7a^{-1} + 6 \\
 & \underline{16a^{-3} + 8a^{-2}} \\
 & \qquad - 14a^{-2} + 5a^{-1} \\
 & \qquad \underline{- 14a^{-2} - 7a^{-1}} \\
 & \qquad\qquad\qquad 12a^{-1} + 6 \\
 & \qquad\qquad\qquad \underline{12a^{-1} + 6}
\end{array}$$

3. Find the square root of $\dfrac{x^2}{y^2} - \dfrac{2y}{x} + \dfrac{y^2}{x^2} - 1 + \dfrac{2x}{y}$.

Arrange in descending powers of x:

$$\begin{array}{rl}
 & x^2y^{-2} + 2xy^{-1} - 1 - 2x^{-1}y + x^{-2}y^2\,(\,xy^{-1} + 1 - x^{-1}y \\
 & \underline{x^2y^{-2}} \\
2xy^{-1} + 1\,) & 2xy^{-1} - 1 - 2x^{-1}y + x^{-2}y^2 \\
 & \underline{2xy^{-1} + 1} \\
2xy^{-1} + 2 - x^{-1}y\,) & -2 - 2x^{-1}y + x^{-2}y^2 \\
 & \underline{-2 - 2x^{-1}y + x^{-2}y^2}
\end{array}$$

Multiply:

4. $a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}}$ by $a^{\frac{1}{3}} - 1 + a^{-\frac{1}{3}}$.

5. $3a^{\frac{3}{5}} - 4a^{\frac{1}{5}} - a^{-\frac{1}{5}}$ by $3a^{\frac{1}{5}} + a^{-\frac{1}{5}} - 6a^{-\frac{3}{5}}$.

Ans. $9a^{\frac{4}{5}} - 9a^{\frac{2}{5}} - 25 + 23a^{-\frac{2}{5}} + 6a^{-\frac{4}{5}}$.

6. $5 + 2x^{2a} + 3x^{-2a}$ by $4x^{a} - 3x^{-a}$.

7. $a^{\frac{3}{2}} - 8a^{-\frac{3}{2}} + 4a^{-\frac{1}{2}} - 2a^{\frac{1}{2}}$ by $4a^{-\frac{3}{2}} + a^{\frac{1}{2}} + 4a^{-\frac{1}{2}}$.

8. $1 - 2\sqrt[3]{x} - 2x^{\frac{1}{2}}$ by $1 - \sqrt[6]{x}$.

9. $2\sqrt[3]{a^5} - a^{\frac{1}{3}} - \frac{3}{a}$ by $2a - 3\sqrt[3]{\frac{1}{a}} - a^{-\frac{5}{3}}$.

Ans. $4a^{\frac{8}{3}} - 8a^{\frac{4}{3}} - 5 + 10a^{-\frac{4}{3}} + 3a^{-\frac{8}{3}}$.

Divide:

10. $21x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$ by $3x^{\frac{1}{3}} + 1$.

Ans. $7x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 1$.

11. $15a - 3a^{\frac{1}{3}} - 2a^{-\frac{1}{3}} + 8a^{-1}$ by $5a^{\frac{2}{3}} + 4$.

12. $16a^{-3} + 6a^{-2} + 5a^{-1} - 6$ by $2a^{-1} - 1$.

13. $5b^{\frac{2}{3}} - 6b^{\frac{1}{3}} - 4b^{-\frac{2}{3}} - 4b^{-\frac{1}{3}} - 5$ by $b^{\frac{1}{6}} - 2b^{-\frac{1}{6}}$.

Ans. $5b^{\frac{1}{2}} + 4b^{\frac{1}{6}} + 3b^{-\frac{1}{6}} + 2b^{-\frac{1}{2}}$.

14. $8c^{-n} - 8c^{n} + 5c^{3n} - 3c^{-3n}$ by $5c^{n} - 3c^{-n}$.

15. $\sqrt[3]{x^2} + 2x^{\frac{1}{3}} - 16x^{-\frac{2}{3}} - \frac{32}{x}$ by $x^{\frac{1}{6}} + 4x^{-\frac{1}{6}} + \frac{4}{\sqrt{x}}$.

Ans. $x^{\frac{1}{2}} - 2x^{\frac{1}{6}} + 4x^{-\frac{1}{6}} - 8x^{-\frac{1}{2}}$.

16. $1 - \sqrt{a} - \frac{2}{a^{-1}} + 2a^2$ by $1 - a^{\frac{1}{2}}$.

17. $4\sqrt[3]{x^2} - 8x^{\frac{1}{3}} - 5 + \frac{10}{\sqrt[3]{x}} + 3x^{-\frac{2}{3}}$ by $2x^{\frac{5}{12}} - \sqrt[12]{x} - \frac{3}{\sqrt[4]{x}}$.

Ans. $2x^{\frac{1}{4}} - 3x^{-\frac{1}{12}} - x^{-\frac{5}{12}}$.

Find the square root of:

18. $9x - 12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1}$.

Ans. $3x^{\frac{1}{2}} - 2 + x^{-\frac{1}{2}}$.

19. $25a^{\frac{4}{3}} + 16 - 30a - 24a^{\frac{1}{3}} + 49a^{\frac{2}{3}}$.

20. $4x^{n} + 9x^{-n} + 28 - 24x^{-\frac{n}{2}} - 16x^{\frac{n}{2}}$.

21. $9x^{-4} - 18x^{-3}\sqrt{y} + \frac{15y}{x^2} - 6\sqrt{\left(\frac{y^3}{x^2}\right)} + y^2$.

Ans. $3x^{-2} - 3x^{-1}y^{\frac{1}{2}} + y$.

CHAPTER XVII.

RADICALS.

241. A **Radical** is the indicated root of a number, as $\sqrt{x}$, $a^{\frac{1}{2}}$, $\sqrt[3]{27}$, $5^{\frac{1}{3}}$, etc. If the indicated root cannot be exactly obtained, the radical is called an *irrational* expression, or **Surd**; otherwise it is called a *rational* expression.

242. The factor standing before the radical is the *coefficient* of the radical. Thus, 5 is the coefficient of $\sqrt{3}$ in the expression $5\sqrt{3}$.

243. **Similar Radicals** are those which do not differ, or differ only in their coefficients. Thus, $\sqrt{a}$, $5\sqrt{a}$, and $b\sqrt{a}$ are *similar* radicals; but $3\sqrt{a}$ and $3\sqrt{b}$, or $3x^{\frac{1}{2}}$ and $3x^{\frac{1}{3}}$, are *dissimilar* radicals.

244. The degree of a radical is denoted by the root index, or by the denominator of the fractional index.

The various operations in radicals are presented under the following cases.

Case I.

245. To reduce a Radical to its Simplest Form.

Note 1. A radical is in its simplest form when it is integral, and contains no factor whose indicated root can be found.

1. Reduce $\sqrt[3]{64a^6b^2}$ to its simplest form.

$$\sqrt[3]{64a^6b^2} = \sqrt[3]{64a^6 \times b^2} = \sqrt[3]{64a^6} \times \sqrt[3]{b^2} = 4a^2\sqrt[3]{b^2}, \text{ Ans.}$$

We first resolve $64a^6b^2$ into two factors, one of which, $64a^6$, is the greatest perfect cube in $64a^6b^2$; then, as the root of the product

is equal to the product of the roots (§ 220, Note 2), we extract the cube root of the perfect cube $64\,a^6$, and annex to this root the factor remaining under the radical. Hence,

Rule.

Resolve the expression under the radical sign into two factors, one of which is the greatest perfect power of the same name as the root. Extract the root of the perfect power, multiply it by the coefficient of the radical, and annex to the result the other factor, with the radical sign before it.

NOTE 2. When the greatest perfect power of the numerical part of the expression cannot be readily determined by inspection, it should be resolved into its prime factors, and these factors treated in just the same way as the literal part of the expression.

Thus,

$$\begin{aligned} & 5\sqrt[3]{648\,a^7\,b^4\,c} \\ &= 5\sqrt[3]{2^3\cdot 3^4\,a^7\,b^4\,c} \\ &= 5\sqrt[3]{2^3\,3^3\,a^6\,b^3}\times\sqrt[3]{3\,a\,b\,c} \\ &= 5\cdot 2\cdot 3\,a^2\sqrt[3]{3\,a\,b\,c} \\ &= 30\,a^2\sqrt[3]{3\,a\,b\,c} \end{aligned}$$

Reduce the following expressions to their simplest form:

2. $\sqrt{8\,a\,x^2}$.
3. $\sqrt[3]{54\,a^7\,b^4}$.
4. $\sqrt{50\,x^3\,y^2}$.
5. $\sqrt[3]{128\,m^3\,n^3}$.
6. $3\sqrt[4]{144\,a^3\,b^2}$.
7. $5\sqrt[3]{250\,a^5\,b^3}$. Ans. $25\,a\,b\sqrt[3]{10\,a^2}$.
8. $3\sqrt[4]{3125\,c^5}$.
9. $\sqrt[3]{-2187}$.
10. $\sqrt[3]{-108\,x^4\,y^3}$ Ans. $-3\,x\,y\sqrt[3]{4\,x}$.
11. $\sqrt[4]{900000}$.
12. $\sqrt{8\,a^{4n}\,x^{6m}}$.
13. $\sqrt[n]{x^{3n}\,y^{2n+5}}$. Ans. $x^3\,y^2\sqrt[n]{y^5}$.
14. $\sqrt{a^3-a^2\,x}$.
15. $(3\,a^7\,b^4-2\,a^3\,b^6)^{\frac{1}{3}}$.
16. $(3\,x^{2m}\,y^{3m}-5\,x^m\,y^{2m})^{\frac{1}{m}}$.

17. $a^{-m} c^3 (a^{mn} c^{2n} - a^{2mn} c^n)^{\frac{1}{n}}$. Ans. $c^4 (c^n - a^{mn})^{\frac{1}{n}}$.

18. $\sqrt{a^3 + 2a^2 b + a b^2}$.

19. $\sqrt[3]{(x+1)^2 (x^2 - 1)}$.

20. $\sqrt[3]{(x^2 - a^2)(x - a)^2}$.

21. $(x - 5)\sqrt{a x^2 + 10 a x + 25 a}$.

22. $(8x^4 y - 24 x^3 y^2 + 24 x^2 y^3 - 8 x y^4)^{\frac{1}{3}}$.
Ans. $2(x - y)\sqrt[3]{x y}$.

If the expression under the radical sign is a fraction, *multiply both terms of the fraction by the least number that will make its denominator a perfect power of the same name as the root, and then remove a factor according to the Rule.*

23. Reduce $\sqrt[3]{\frac{3}{4}}$ to its simplest form.

$$\sqrt[3]{\tfrac{3}{4}} = \sqrt[3]{\tfrac{6}{8}} = \sqrt[3]{\tfrac{1}{8}}\,\sqrt[3]{6} = \tfrac{1}{2}\sqrt[3]{6} \text{ Ans.}$$

24. Reduce $\frac{2}{7}\sqrt[3]{\frac{864}{1875\,b^2}}$ to its simplest form.

$$\frac{2}{7}\sqrt[3]{\frac{864}{1875\,b^2}} = \frac{2}{7}\sqrt[3]{\frac{2^5 \cdot 3^3}{5^4 \cdot 3\,b^2}} = \frac{2}{7}\sqrt[3]{\frac{2^3}{5^6\,b^3}}\,\sqrt[3]{2^2 \cdot 5^2 \cdot 3^2\,b}$$

$$= \frac{2 \cdot 2}{7 \cdot 5^2 b}\sqrt[3]{2^2 \cdot 5^2 \cdot 3^2\,b} = \frac{4}{175\,b}\sqrt[3]{900\,b} \text{ Ans.}$$

Reduce the following to their simplest forms:

25. $\frac{1}{3}\sqrt{\frac{1}{2}}$.

26. $3\sqrt[3]{\frac{2}{3}}$.

27. $\frac{2}{5}\sqrt{\frac{4}{5}}$.

28. $\frac{4}{11}\sqrt{\frac{77}{8}}$. Ans. $\frac{1}{11}\sqrt{154}$.

29. $5\sqrt{7.728}$.

30. $\frac{3ab}{2c}\sqrt{\frac{20\,c^2}{9\,a^2 b}}$.

31. $\frac{2a}{3x}\sqrt[3]{\frac{27\,x^4}{a^2}}$.

32. $\frac{2a}{b}\sqrt[4]{\frac{b^4}{8\,a^3}}$.

33. $a\sqrt[n]{\frac{b^2}{a^{n-2}}}$. Ans. $\sqrt[n]{a^2 b^2}$.

34. $\frac{a}{b}\sqrt[n]{\frac{b^{n+1}}{a^{n-1}}}$.

35. $\frac{y}{x^n}\sqrt{\frac{x^{2n+1}}{y^3}}$. Ans. $\frac{1}{y}\sqrt{xy}$.

36. $(x+y)\sqrt{\frac{x-y}{x+y}}$.

37. $\frac{ax}{a-x}\sqrt{\frac{a^2-x^2}{a^2x^2}}$.

38. $\frac{c}{x^2-a^2}\sqrt{\frac{x^3-2ax^2+a^2x}{c^3}}$. Ans. $\frac{1}{c(x+a)}\sqrt{cx}$.

246. Conversely, the coefficient of a radical can be placed under the radical sign, *by involving it to a power of the same degree as the root indicated by the radical sign, multiplying it by the radical, and placing the given radical sign over the product.*

NOTE. Likewise a whole or fractional number may be expressed in the form of a radical, by raising the number to a power denoted by the degree of the radical, and placing the result under the corresponding radical sign.

In the following examples, place the coefficient under the radical sign.

1. $4\sqrt{3x}$.

$$4\sqrt{3x} = \sqrt{16}\sqrt{3x} = \sqrt{48x} \text{ Ans.}$$

2. $5\sqrt[3]{2x^2y}$.

3. $3x\sqrt{xy}$.

4. $2ab\sqrt{1-3b}$.

5. $(x+y)\sqrt[3]{\frac{x}{x+y}}$.

6. $\frac{x-y}{x+y}\left(\frac{x^2+xy}{x^2-2xy+y^2}\right)^{\frac{1}{2}}$.

Express as radicals of the 4th and nth degrees:

7. 2. 8. $\frac{1}{3}$. 9. x^2. 10. x^a.

Which is greater:

11. $5\sqrt{3}$ or $3\sqrt{5}$?

12. $2\sqrt{6}$ or $3\sqrt{3}$?

13. $3\sqrt[3]{5}$ or $2\sqrt[3]{17}$?

14. $5\sqrt[3]{2}$ or $4\sqrt[3]{4}$?

15. Arrange in order of magnitude $\frac{1}{2}\sqrt{2}$, $\frac{1}{3}\sqrt{3}$, $\frac{1}{4}\sqrt{4}$.

Case II.

247. To reduce Radicals of different Degrees to equivalent ones of the same Degree.

1. Reduce $\sqrt{a}$ and $\sqrt[3]{b}$ to equivalent radicals of the same degree.

$$a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}$$
$$b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}$$

In this case we write the radicals with their fractional indices; and then, as the denominator is the index of the root, in order that the two radicals may have the same root index, we reduce the fractional indices to equivalent ones having a common denominator. It is evident that we have not changed the values of the given radicals by the process. Hence,

Rule.

Reduce the fractional indices to equivalent ones having a common denominator; involve each number to the power denoted by the numerator of the reduced index, and indicate the root denoted by the denominator.

Reduce the following to equivalent radicals having a common index:

2. $\sqrt{3}$, $\sqrt[3]{5}$.

$$\left.\begin{array}{l} 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = 27^{\frac{1}{6}} = \sqrt[6]{27} \\ 5^{\frac{1}{3}} = 5^{\frac{2}{6}} = 25^{\frac{1}{6}} = \sqrt[6]{25} \end{array}\right\} \text{Ans.}$$

3. $\sqrt{\frac{1}{3}}$, $\sqrt[3]{\frac{1}{5}}$.

4. $\sqrt{\frac{a}{b}}$, $\sqrt[4]{\frac{c}{d}}$.

5. $\sqrt{x+y}$, $\sqrt[3]{x-y}$, $\sqrt[4]{x}$.

6. $\sqrt{3}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$.

7. $\sqrt[m]{x}$, $\sqrt[n]{y}$.

8. $\sqrt[m]{x^n}$, $\sqrt[n]{x^m}$.

9. $\sqrt[m]{a^n}$, $\sqrt[n]{b^r}$, $\sqrt[r]{c^m}$. Ans. $\sqrt[mnr]{a^{n^2r}}$, $\sqrt[mnr]{b^{mr^2}}$, $\sqrt[mnr]{c^{m^2n}}$.

10. Which is greater, $\sqrt{2}$ or $\sqrt[3]{3}$?

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{8}$$
$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{9}$$

Ans. $\sqrt[3]{3}$.

Which is greater:

11. $\sqrt{2}$ or $\sqrt[5]{5}$? 12. $\sqrt[3]{2}$ or $\sqrt[4]{3}$? 13. $\sqrt[4]{4}$ or $\sqrt[3]{3}$?

Arrange in the order of their magnitude:

14. $\sqrt{3}$, $\sqrt[3]{6}$, and $\sqrt[4]{12}$. 15. $\sqrt{5}$, $\sqrt[4]{10}$, and $\sqrt[6]{15}$.

CASE III.

248. Addition and Subtraction of Radicals.

1. Find the sum of $2\sqrt{a}$ and $3\sqrt{a}$.

It is evident that 2 times the $\sqrt{a}$ and 3 times the $\sqrt{a}$ make 5 times the $\sqrt{a}$. Ans. $5\sqrt{a}$.

2. From $7\sqrt{a}$ take $4\sqrt{a}$.

It is evident that 7 times the $\sqrt{a}$ minus 4 times the $\sqrt{a}$ is 3 times the $\sqrt{a}$. Ans. $3\sqrt{a}$.

3. Find the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

$$\begin{aligned} \sqrt[3]{500} &= 5\sqrt[3]{4} \\ \sqrt[3]{108} &= 3\sqrt[3]{4} \\ \hline \text{Sum} &= 8\sqrt[3]{4} \end{aligned}$$

In this case we make the radical parts similar by reducing them to their simplest form (§ 245), and then add their coefficients as in Example 1.

4. From $\sqrt{200}$ take $\sqrt{128}$.

$$\begin{aligned} \sqrt{200} &= 10\sqrt{2} \\ \sqrt{128} &= 8\sqrt{2} \\ \hline \text{Remainder} &= 2\sqrt{2} \end{aligned}$$

We make the radical parts similar by reducing them to their simplest form (§ 245). And $8\sqrt{2}$ taken from $10\sqrt{2}$ evidently leaves $2\sqrt{2}$.

5. Simplify $\sqrt{50}+\sqrt{54}-\sqrt{24}+2\sqrt{\frac{3}{2}}$.

$$\begin{aligned} \sqrt{50} &= \sqrt{25\cdot 2} &&= 5\sqrt{2} \\ \sqrt{54} &= \sqrt{9\cdot 6} &&= 3\sqrt{6} \\ -\sqrt{24} &= -\sqrt{4\cdot 6} &&= -2\sqrt{6} \\ 2\sqrt{\tfrac{3}{2}} &= 2\sqrt{\tfrac{3\cdot 2}{2^2}} &&= \sqrt{6} \\ \hline && \text{Algebraic sum} &= 2\sqrt{6}+5\sqrt{2} \end{aligned}$$

From the above examples we deduce the following

Rule.

Reduce each radical to its simplest form; then find the algebraic sum of the coefficients of similar radicals, prefix it to the common radical, and indicate the addition or subtraction of the dissimilar radicals.

Simplify the following:

6. $\sqrt{32} + \sqrt{50}$.

7. $3\sqrt{12\,xy} + 5\sqrt{75\,xy}$.

8. $\sqrt{125} - \sqrt{45}$.

9. $\sqrt[3]{54} - \sqrt[3]{16}$.

10. $3\sqrt[3]{192\,a^3} + 2\sqrt[3]{81\,a^3}$.

11. $2\sqrt[4]{405\,x^4} + 3\sqrt[4]{1280\,y^4}$.

12. $\sqrt[3]{\frac{1}{16}} + \sqrt[3]{\frac{1}{54}}$.

13. $5\sqrt{24} - 2\sqrt{54} - \sqrt{6}$. Ans. $3\sqrt{6}$.

14. $\sqrt[3]{81} - 7\sqrt[3]{192} + 4\sqrt[3]{648}$.

15. $\sqrt{44} - 5\sqrt{176} + 2\sqrt{99}$. Ans. $-12\sqrt{11}$.

16. $4\sqrt{128} + 4\sqrt{75} - 5\sqrt{162}$.

17. $3\sqrt[4]{162} - 7\sqrt[4]{32} + \sqrt[4]{1250}$.

18. $5\sqrt[3]{-54} - 2\sqrt[3]{-16} + 4\sqrt[3]{686}$. Ans. $17\sqrt[3]{2}$.

19. $2\sqrt{363} - 5\,(243)^{\frac{1}{2}} + (192)^{\frac{1}{2}}$.

20. $\sqrt{252} - \sqrt{294} - \dfrac{48}{\sqrt{6}}$. Ans. $6\sqrt{7} - 15\sqrt{6}$.

21. $3\sqrt{147} - \frac{7}{3}\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{27}}$.

22. $\dfrac{\sqrt[3]{72}}{18} - \frac{1}{3}\sqrt[3]{\frac{1}{3}} + 6\sqrt[3]{21\frac{1}{3}}$ Ans. $8\sqrt[3]{9}$.

23. $\sqrt{20} + 3\sqrt{\frac{4}{5}} + 3\,(\frac{5}{9})^{-\frac{1}{2}} - (\frac{25}{81})^{\frac{1}{4}}$.

24. $(\frac{1}{20})^{-\frac{1}{2}} + \dfrac{3}{\sqrt{5}} + 4\sqrt{125} + 7\,(\frac{9}{5})^{\frac{1}{2}} + 3\,(\frac{80}{9})^{-\frac{1}{2}} + \sqrt[4]{\frac{25}{16}}$.

Ans. $27\frac{3}{4}\sqrt{5}$.

25. $2a\sqrt{c^2x-c^2y}-3c\sqrt{a^2x-a^2y}+5\sqrt{a^2c^2x-a^2c^2y}$.

26. $(a^3+2a^2b+ab^2)^{\frac{1}{2}}-(a^3-2a^2b+ab^2)^{\frac{1}{2}}$.

27. $5(ax^3-bx^3)^{\frac{1}{3}}-3(bx^3-ax^3)^{\frac{1}{3}}$. Ans. $8x(a-b)^{\frac{1}{3}}$.

28. $\sqrt{\dfrac{x+y}{x-y}}+\sqrt{\dfrac{x-y}{x+y}}$.

29. $\sqrt{\dfrac{a^2(x+1)}{x-1}}+b\sqrt{\dfrac{x-1}{x+1}}-\sqrt{b^2(x^2-1)(x-1)^{-2}}$.

Ans. $\dfrac{ax+a-2b}{x^2-1}\sqrt{x^2-1}$.

Case IV.

249. Multiplication of Radicals.

1. Multiply $2\sqrt{a}$ by $5\sqrt{b}$.

$$2\sqrt{a}\times5\sqrt{b}=2\times5\times\sqrt{a}\times\sqrt{b}=10\sqrt{ab}$$

As it makes no difference in what order the factors are taken (§ 66), we unite in one product the numerical coefficients; and $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ (§ 220, Note 2).

2. Multiply $3\sqrt{2a}$ by $4\sqrt[3]{3ab}$.

$$\begin{aligned} 3\sqrt{2a} &= 3\sqrt[6]{8a^3} \\ 4\sqrt[3]{3ab} &= 4\sqrt[6]{9a^2b^2} \\ \hline \text{Product} &= 12\sqrt[6]{72a^5b^2} \end{aligned}$$

We reduce the radical parts to equivalent radicals of the same degree (§ 247), and then multiply as in the preceding example.

From these examples we deduce the following

Rule.

Reduce the radical parts, if necessary, to equivalent radicals of the same degree, and to the product of the radical parts placed under the common radical sign prefix the product of their coefficients.

Find the value of:

3. $2\sqrt{15} \times 3\sqrt{5}$.

4. $8\sqrt{12} \times 3\sqrt{24}$.

5. $5\sqrt[3]{128} \times 2\sqrt[3]{432}$.

6. $5\sqrt{32} \times \sqrt{48} \times 2\sqrt{54}$. Ans. 2880.

7. $a\sqrt{b^3} \times b^2\sqrt{a}$.

8. $5\sqrt[3]{2} \times 2\sqrt{5}$. Ans. $10\sqrt[6]{500}$.

9. $\sqrt{2} \times \sqrt[3]{3} \times \sqrt[4]{5}$.

10. $\sqrt[m]{a} \times \sqrt[n]{b}$. Ans. $\sqrt[mn]{a^n b^m}$.

11. $\sqrt{2pq} \times \sqrt[3]{4q^2r} \times \sqrt[4]{3p^3r}$.

12. $3\sqrt{8} \times 2\sqrt[3]{6} \times 3\sqrt[4]{54}$.

13. $2\sqrt{24} \times 3\sqrt[4]{18} \times 4\sqrt[6]{24}$. Ans. $288\sqrt[12]{72}$.

14. $2\sqrt{\frac{2}{3}} \times 3\sqrt{\frac{3}{2}}$.

15. $\dfrac{3}{x}\sqrt{\dfrac{a^2}{x}} \times \dfrac{4}{3}\sqrt{\dfrac{x^3}{2a^4}}$.

16. $\dfrac{3\sqrt{11}}{2\sqrt{98}} \times \dfrac{7\sqrt{22}}{5}$. Ans. 3 3.

17. $\sqrt{\dfrac{ax^3}{(a+x)^4}} \times \sqrt{\dfrac{d(a^2-x^2)}{x^6}} \times \sqrt[4]{\dfrac{a^3c^2}{(a-x)^3}}$.

18. Multiply $2\sqrt{5} + 3\sqrt{x}$ by $\sqrt{5} - \sqrt{x}$.

$$\begin{array}{llll}
2\sqrt{5} & + 3\sqrt{x} & & \\
\sqrt{5} & - \sqrt{x} & & \\
\hline
10 & + 3\sqrt{5x} & & \\
 & - 2\sqrt{5x} & - 3x & \\
\hline
10 & + \sqrt{5x} & - 3x & \text{Ans.}
\end{array}$$

19. $(2\sqrt{x} - 5) \times 3\sqrt{x}$.

20. $(\sqrt{a} + \sqrt{b}) \times \sqrt{ab}$.

21. $(3\sqrt{5} - 4\sqrt{2})(2\sqrt{5} + 3\sqrt{2})$. Ans. $6 + \sqrt{10}$.

22. $(\sqrt{2} + \sqrt{3} - \sqrt{5})(\sqrt{2} + \sqrt{3} + \sqrt{5})$.

23. $(\sqrt{5} + 3\sqrt{2} + \sqrt{7})(\sqrt{5} + 3\sqrt{2} - \sqrt{7})$.
Ans. $16 + 6\sqrt{10}$.

24. $(2\sqrt{3} + 3\sqrt{2})^2$.

25. $(\sqrt{x+a} - \sqrt{x-a})(\sqrt{x+a})$.

26. $(\sqrt{a+x} - \sqrt{a-x})^2$. Ans. $2a - 2\sqrt{a^2 - x^2}$.

27. $(3\sqrt{7} + 5\sqrt{11})(3\sqrt{7} - 5\sqrt{11})$.

28. $(3\sqrt{a} + \sqrt{x - 9a})(3\sqrt{a} - \sqrt{x - 9a})$.
Ans. $18a - x$.

29. $(3\sqrt{a^2 + b^2} - 2\sqrt{a^2 - b^2})^2$.

30. $(\sqrt{a+x} + \sqrt{a-x})(\sqrt{a+x} - \sqrt{a-x})$.

31. $(3\sqrt{2} + 2\sqrt{3})(2\sqrt{3} - 3\sqrt{2})(3\sqrt{3} + 2\sqrt{2})$.
Ans. $-18\sqrt{3} - 12\sqrt{2}$.

Case V.

250. Division of Radicals.

1. Divide $15\sqrt{6x^2}$ by $5\sqrt{2x}$.

$$\frac{15\sqrt{6x^2}}{5\sqrt{2x}} = 3\sqrt{3x}$$

As division is finding a quotient which, multiplied by the divisor, will produce the dividend, the coefficient of the quotient must be a number which, multiplied by 5, will give 15, the coefficient of the dividend, that is, 3; and the radical part of the quotient must be a number which, multiplied by $\sqrt{2x}$, will give $\sqrt{6x^2}$, that is, $\sqrt{3x}$; the quotient required, therefore, is $3\sqrt{3x}$.

2. Divide $8\sqrt{6a}$ by $4\sqrt[3]{3a}$.

$$\frac{8\sqrt{6a}}{4\sqrt[3]{3a}} = \frac{8\sqrt[6]{216a^3}}{4\sqrt[6]{9a^2}} = 2\sqrt[6]{24a} \quad \text{Ans.}$$

We reduce the radical parts to equivalent radicals of the same degree (§ 247), and then divide as in the preceding example.

From these examples we deduce the following

Rule.

Reduce the radical parts, if necessary, to equivalent radicals of the same degree, and to the quotient of the radical parts placed under the common radical sign prefix the quotient of their coefficients.

Find the value of:

3. $5\sqrt{27} \div 3\sqrt{24}$.

4. $21\sqrt{384} \div 8\sqrt{98}$.
   Ans. $3\sqrt{3}$.

5. $-13\sqrt{125} \div 5\sqrt{65}$.

6. $4c^2\sqrt{6a^3} \div 3c\sqrt[3]{8a^5}$.

7. $5 \div \sqrt{5}$.

8. $\dfrac{3\sqrt{48}}{5\sqrt{112}} \div \dfrac{6\sqrt{84}}{\sqrt{392}}$.
   Ans. $\frac{1}{10}\sqrt{2}$.

9. $6\sqrt[6]{1.728} \div 3\sqrt[4]{1.44}$.

10. $\sqrt[4]{90000} \div \sqrt[5]{32}$.
   Ans. $5\sqrt{3}$.

11. $\dfrac{2}{7}\sqrt{\dfrac{a}{b}} \div \dfrac{3}{5}\sqrt[3]{\dfrac{b}{a}}$.

12. $\dfrac{3}{a-b}\sqrt{\dfrac{2x}{a-b}} \div \sqrt{\dfrac{18x^3}{(a-b)^5}}$.
   Ans. $\dfrac{a-b}{x}$.

Case VI.

251. Involution and Evolution of Radicals.

1. Find the cube of $2\sqrt{a}$.

$$(2\sqrt{a})^3 = 2\sqrt{a} \times 2\sqrt{a} \times 2\sqrt{a}$$
$$= 8\sqrt{a^3} = 8a\sqrt{a}$$

In accordance with the definition of involution, we take the number three times as a factor. By Art. 249 the product is $8a\sqrt{a}$.

2. Find the cube root of $27\,x^3\sqrt{a\,b}$.

$$\sqrt[3]{27\,x^3\sqrt{a\,b}} = 3\,x\,\sqrt[6]{a\,b}$$

As the root of the product is equal to the product of the roots (§ 238), we prefix to the cube root of the radical part the cube root of the rational part. The cube root of the radical part must be a number which, taken three times as a factor, will produce $\sqrt{a\,b}$; that is, $\sqrt[6]{a\,b}$.

3. Find the square of $3\sqrt[3]{x}$.

$$(3\sqrt[3]{x})^2 = (3\,x^{\frac{1}{3}})^2 = 9\,x^{\frac{2}{3}}$$

In this case we have used the fractional exponent, and found the square of the given number by multiplying its exponent by the index of the required power, according to Art. 236.

Note 1. Dividing the index of the root is the same as multiplying the fractional exponent. Thus, the square of $\sqrt[6]{a}$ is $\sqrt[3]{a}$; for $(a^{\frac{1}{6}})^2 = a^{\frac{1}{3}}$, or $\sqrt[3]{a}$.

4. Find the fourth root of $\sqrt{x}$.

$$\sqrt[4]{\sqrt{x}} = (x^{\frac{1}{2}})^{\frac{1}{4}} = x^{\frac{1}{8}},\ \text{or}\ \sqrt[8]{x}$$

In this case we have used the fractional exponent, and found the fourth root by dividing the exponent of the given number by the index of the required root, according to Art. 236.

Note 2. Multiplying the index of the root is the same as dividing the fractional exponent. Thus, the square root of $\sqrt[5]{b}$ is $\sqrt[10]{b}$; for $(b^{\frac{1}{5}})^{\frac{1}{2}} = b^{\frac{1}{10}}$, or $\sqrt[10]{b}$.

From these examples we deduce for Involution and Evolution the following

Rule.

I. *Involve or evolve the radical as if it were rational, and, placing it under its proper radical sign, prefix the required power or root of its coefficient.*

II. *A radical can be involved or evolved by multiplying or dividing its fractional exponent by the index of the required power or root.*

Perform the operations indicated in the following examples:

5. $(4x\sqrt{y})^3$.
6. $(3a\sqrt[3]{x})^2$.
7. $\sqrt{8a^2\sqrt{3}}$.
8. $\sqrt[3]{54\sqrt{2}}$.
9. $\sqrt[3]{(x^2 z\sqrt{xyz})}$.
10. $\sqrt[4]{(b^2 c d^3 \sqrt[3]{a^2 c d})}$.
11. $\sqrt[3]{(16\sqrt{2a})}$. Ans. $2\sqrt[6]{8a}$.
12. $\sqrt{\{2\sqrt[3]{(4\sqrt[5]{2b})}\}}$.

Case VII.

Rationalization.

252. A fraction having a radical for its denominator can be changed to an equivalent fraction with a rational denominator. This is called *rationalizing* the denominator of the fraction.

1st. *When the denominator is a monomial.*

1. Rationalize the denominator of $\frac{5}{\sqrt{2}}$.

$$\frac{5}{\sqrt{2}} = \frac{5\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} = \frac{5\sqrt{2}}{2} \quad \text{Ans.}$$

2. Rationalize the denominator of $\frac{7}{\sqrt[3]{3}}$.

$$\frac{7}{\sqrt[3]{3}} = \frac{7\times 3^{\frac{2}{3}}}{3^{\frac{1}{3}}\times 3^{\frac{2}{3}}} = \frac{7\sqrt[3]{9}}{3} \quad \text{Ans.}$$

From these examples it will be seen that, to rationalize the denominator of a fraction when that denominator is a monomial, we *multiply both terms of the fraction by the number that is under the radical in the denominator with an index equal to* 1 *minus the fractional index of this number.*

Rationalize the denominators of the following fractions:

3. $\frac{5}{\sqrt{3}}$. 4. $\frac{\sqrt{3x}}{\sqrt{2a}}$. 5. $\frac{4}{\sqrt[3]{2}}$. 6. $\frac{8a}{\sqrt[4]{2}}$.

7. $\dfrac{3}{\sqrt[3]{4}}$. 8. $\dfrac{1}{\sqrt[4]{8}}$. 9. $\dfrac{5}{\sqrt[3]{5}}$. 10. $\dfrac{1}{\sqrt[4]{3}}$.

11. Find the square root of $\dfrac{5}{8}$.

$$\sqrt{\frac{5}{8}}=\frac{\sqrt{5}}{\sqrt{8}}=\frac{\sqrt{5}}{2\sqrt{2}}=\frac{\sqrt{5}\times\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}=\frac{\sqrt{10}}{4}$$

$$\frac{\sqrt{10}}{4}=\tfrac{1}{4}(3.1623-)=0.7905+ \quad \text{Ans.}$$

By this method find the values of the following:

12. $\sqrt{\frac{2}{3}}$. 13. $\sqrt[3]{\frac{3}{5}}$. 14. $\sqrt{\frac{1}{3}}$. 15. $\sqrt[3]{\frac{1}{3}}$.

2d. *When the denominator is a binomial involving radicals of the second degree only.*

16. Rationalize the denominator of $\dfrac{5}{\sqrt{3}+\sqrt{2}}$.

$$\frac{5}{\sqrt{3}+\sqrt{2}}=\frac{5(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}=\frac{5\sqrt{3}-5\sqrt{2}}{3-2}$$
$$=5\sqrt{3}-5\sqrt{2} \quad \text{Ans.}$$

17. Rationalize the denominator of $\dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$.

$$\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}=\frac{(\sqrt{7}+\sqrt{3})(\sqrt{7}+\sqrt{3})}{(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})}=\frac{7+2\sqrt{21}+3}{7-3}$$
$$=\frac{10+2\sqrt{21}}{4}=\frac{5}{2}+\tfrac{1}{2}\sqrt{21} \quad \text{Ans.}$$

From these examples it will be seen that, when the denominator of a fraction is a binomial involving radicals of the second degree only, the denominator can be rationalized *by multiplying both terms of the fraction by the denominator, with the opposite sign between the terms.*

Rationalize the denominators in the following fractions:

18. $\dfrac{5}{\sqrt{4}-\sqrt{3}}$. 19. $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$.

20. $\dfrac{3}{3+\sqrt{2}}$.

21. $\dfrac{3}{3-\sqrt{2}}$.

22. $\dfrac{2-\sqrt{2}}{2+\sqrt{2}}$.

23. $\dfrac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}}$.

24. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

25. $\dfrac{3\sqrt{6}+\sqrt{3}}{2\sqrt{6}-\sqrt{3}}$.

26. $\dfrac{\sqrt{x+1}+3}{\sqrt{x+1}-2}$.

27. $\dfrac{\sqrt{x-2}+\sqrt{x}}{\sqrt{x-2}-\sqrt{x}}$.

28. $\dfrac{\sqrt{a^2+1}-a}{\sqrt{a^2+1}+a}$.

29. $\dfrac{a+\sqrt{a^2+3}}{a-\sqrt{a^2+3}}$.

30. $\dfrac{\sqrt{a-b}-\sqrt{a+b}}{\sqrt{a-b}+\sqrt{a+b}}$.

31. $\dfrac{\sqrt{2x-1}+\sqrt{x+1}}{\sqrt{2x-1}-\sqrt{x+1}}$.

32. $\dfrac{10}{\sqrt{8}+\sqrt{108}+\sqrt{18}-\sqrt{12}+\sqrt{3}}$.

33. $\dfrac{2}{\sqrt{3}-1}$.

34. $\dfrac{3}{\sqrt{2}+\sqrt{3}}$.

35. $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

3d. *When the denominator is a trinomial involving radicals of the second degree only.*

36. Rationalize the denominator of $\dfrac{2}{\sqrt{7}+\sqrt{3}+\sqrt{2}}$.

$$\frac{2}{\sqrt{7}+\sqrt{3}+\sqrt{2}}=\frac{2(\sqrt{7}+\sqrt{3}-\sqrt{2})}{(\sqrt{7}+\sqrt{3}+\sqrt{2})(\sqrt{7}+\sqrt{3}-\sqrt{2})}$$

$$=\frac{2(\sqrt{7}+\sqrt{3}-\sqrt{2})}{(\sqrt{7}+\sqrt{3})^2-(\sqrt{2})^2}=\frac{2(\sqrt{7}+\sqrt{3}-\sqrt{2})}{8-2\sqrt{21}}$$

$$=\frac{\sqrt{7}+\sqrt{3}-\sqrt{2}}{4-\sqrt{21}}=\frac{(\sqrt{7}+\sqrt{3}-\sqrt{2})(4+\sqrt{21})}{(4-\sqrt{21})(4+\sqrt{21})}$$

$$=\frac{7\sqrt{7}+11\sqrt{3}-4\sqrt{2}-\sqrt{42}}{5}\quad\text{Ans.}$$

From this example it will be seen that in this case rationalization is effected by twice applying the principle stated in the preceding article.

Rationalize the denominators of the following fractions:

37. $\dfrac{4}{\sqrt{5}+\sqrt{3}-\sqrt{2}}$.

38. $\dfrac{\sqrt{a}-\sqrt{b}-\sqrt{c}}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$.

39. $\dfrac{3}{\sqrt{2}-\sqrt{6}-\sqrt{7}}$.

40. $\dfrac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{a}-\sqrt{b}-\sqrt{c}}$.

IMAGINARIES.

253. An **Imaginary Number** is an indicated even root of a negative number; as, $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-a}$. In distinction from imaginaries, all other numbers (§ 39) are called *real*.

254. The symbol $\sqrt{-1}$ is called the *Imaginary Unit*, and may be defined as an expression which when multiplied by itself produces -1. Therefore,

$$\sqrt{-1}\times\sqrt{-1}=(\sqrt{-1})^2=-1$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2\ \sqrt{-1}\ =-\sqrt{-1}$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2\,(\sqrt{-1})^2=(-1)\,(-1)=1$$

Thus, the first four powers of $\sqrt{-1}$ are, in order, $\sqrt{-1}$, -1, $-\sqrt{-1}$, 1; and if we develop the higher powers, we shall find this order continually repeated.

255. As $(\sqrt{-1})^2=-1$, therefore, $(\sqrt{-1})\,(\sqrt{-1})\,a=-a$. Now, if a represents a line measured in one direction, $-a$ represents a line of equal length measured in the opposite direction (§ 39); that is, a line which could be obtained by turning on the zero point as a centre, the first line through two right angles. Hence, the operation by $\sqrt{-1}$ must indicate the revolution of the line a through *one* right angle.

Now all imaginary numbers may be reduced to the form $a\sqrt{-1}$, and hence may be represented by lengths set off on a straight line passing through the zero point at *right angles* to the line of real numbers. That is, if a represents a certain number of units on the line of real numbers, $a\sqrt{-1}$ will represent the same number of units on a line through the zero point at right angles to the first line.

ADDITION AND SUBTRACTION OF IMAGINARIES.

256. Addition and Subtraction of Imaginaries do not differ materially from Addition and Subtraction of ordinary Radicals.

1. Add $\sqrt{-9}$ and $\sqrt{-25}$.

$$\begin{aligned} \sqrt{-9} &= \sqrt{9(-1)} = 3\sqrt{-1} \\ \sqrt{-25} &= \sqrt{25(-1)} = 5\sqrt{-1} \\ \hline & \phantom{=\sqrt{25(-1)}} = 8\sqrt{-1} \end{aligned}$$

2. Simplify $\sqrt{-49} - 2\sqrt{-4} + \sqrt[4]{-16}$.

$$\begin{aligned} \sqrt{-49} &= \sqrt{49(-1)} = 7\sqrt{-1} \\ -2\sqrt{-4} &= -2\sqrt{4(-1)} = -4\sqrt{-1} \\ \sqrt[4]{-16} &= \sqrt[4]{16(-1)} = 4\sqrt{-1} \\ \hline & \phantom{=\sqrt{16(-1)}} = 7\sqrt{-1} \end{aligned}$$

Rule.

Reduce the imaginaries to the form $a\sqrt{-1}$, *and then affix the imaginary unit to the sum of the coefficients of these units.*

Simplify the following:

3. $3\sqrt{-9} - 2\sqrt[6]{-4096}$. Ans. $\sqrt{-1}$.

4. $5\sqrt{-25} + 3\sqrt{-81} - 7\sqrt{-49}$.

5. $6a\sqrt{-25} - 5a\sqrt{-9} + 3a\sqrt{-4}$. Ans. $21a\sqrt{-1}$.

6. $(5a + b\sqrt{-1}) - (3a + 5b\sqrt{-1}) + (8 - 9b\sqrt{-1})$.

7. $5a\sqrt{-1} + (2a + 6b\sqrt{-1}) - (3a - 2b\sqrt{-1}) + 7a\sqrt{-1}$.

8. $(a + b\sqrt{-1}) + (d + c\sqrt{-1})$.

9. $(a - b\sqrt{-1}) + (a + b)\sqrt{-1}$.

10. $(\sqrt{2} + 3\sqrt{-1}) + (\sqrt{2} - 3\sqrt{-1})$.

MULTIPLICATION AND DIVISION OF IMAGINARIES.

257. As Imaginaries are Radicals, the rules already given for Multiplication and Division of Radicals (§§ 249, 250) might seem sufficient for the Multiplication and Division of Imaginaries. The only difficulty is in the signs. For example, $\sqrt{-3} \times \sqrt{-2}$ is not $\sqrt{6}$, unless we call this square root $-$; the square root of this particular 6, the product of -3 and -2, is not ambiguous, it is minus.

As $(\sqrt{-1})(\sqrt{-1}) = -1$, $(\sqrt{-2})(\sqrt{-2}) = -2$, so $(\sqrt{-2})(\sqrt{-8}) =$ not $\sqrt{16}$ (unless we call it $-$), not $\sqrt{-16}$, but $-\sqrt{16}$, or -4. The true result can be secured by reducing the imaginaries to the form $a\sqrt{-1}$.

1. Multiply $\sqrt{-3}$ by $\sqrt{-5}$.

$$(\sqrt{-3})(\sqrt{-5}) = \sqrt{3}\sqrt{-1}\sqrt{5}\sqrt{-1} = \sqrt{3}\sqrt{5}(\sqrt{-1})^2$$
$$= \sqrt{15}(-1) = -\sqrt{15} \quad \text{Ans.}$$

2. Divide $-\sqrt{15}$ by $\sqrt{-5}$.

$$(\sqrt{-15}) \div \sqrt{-5} = (-1\sqrt{15}) \div (\sqrt{5}\sqrt{-1})$$
$$= \frac{\sqrt{15}}{\sqrt{5}} \times \frac{-1}{\sqrt{-1}} = \sqrt{3}\sqrt{-1} = \sqrt{-3} \quad \text{Ans.}$$

3. Multiply $-\sqrt{10}$ by $\sqrt{-10}$.

$$(-\sqrt{10})(\sqrt{-10}) = (-1\sqrt{10})(\sqrt{-1}\sqrt{10}) = (-1)(\sqrt{-1})(\sqrt{10})^2$$
$$= -10\sqrt{-1} \quad \text{Ans.}$$

4. Divide $\sqrt{-15}$ by $\sqrt{-3}$.

$$(\sqrt{-15}) \div (\sqrt{-3}) = (\sqrt{15}\sqrt{-1}) \div (\sqrt{3}\sqrt{-1})$$
$$= \frac{\sqrt{15}}{\sqrt{3}} \times \frac{\sqrt{-1}}{\sqrt{-1}} = \sqrt{5} \quad \text{Ans.}$$

Rule.

Reduce the imaginary terms to the form $a\sqrt{-1}$. *Multiply or divide the real parts and the imaginary units separately; the product or quotient of the two results will be the answer required.*

NOTE 1. The treatment of signs in roots may be better understood by considering the following:

If we change -3 to $+3$, we change from minus to plus; if we change $(-3)^2$ to $(+3)^2$, we have changed the signs of *two* factors, and the result is no change at all; $(-3)^2$ and $(+3)^2$ both give $+9$. So $(a-b)^2 = (b-a)^2$. In changing $(-3)^3$ to $(+3)^3$ we change the signs of three factors, and the result is a change from minus to plus; and so on. That is, changing the sign of the number to be raised to an odd power changes the result; to an even power does not change the result; and changing the sign of a number to be raised to the first power is *one* change, to the second power *two* changes, to the third power *three* changes, and so on; that is, the number of changes is equal to the index of the power. In roots then, that is, when the index is a fraction, the same law would in the square root make *half* a change, in cube root a *third* of a change, and so on.

Examples 3 and 4 may be more readily worked by the ordinary method, thus:

3. $(-\sqrt{10}) \times (\sqrt{-10} = -\sqrt{-100} = -10\sqrt{-1}$ Ans.

4. $(\sqrt{-15}) \div \sqrt{-3} = \sqrt{5}$ Ans.

NOTE 2. A departure from the ordinary method will only be necessary then, in multiplication, when *both* terms are imaginary, and in division when *one* is imaginary and the other real.

258. The sum of a real and an imaginary is called a *Complex Number;* as, $a + b\sqrt{-1}$.

259. Two complex numbers which differ only in the sign of their imaginary part are said to be *conjugate;* as,

$$-3-2\sqrt{-1} \text{ and } -3+2\sqrt{-1};$$

$$a+b\sqrt{-1} \text{ and } a-b\sqrt{-1}.$$

260. The sum, and the product, of two conjugate complex numbers are real.

For $a+b\sqrt{-1}+a-b\sqrt{-1}=2a;$

and $(a+b\sqrt{-1})(a-b\sqrt{-1})=a^2-(-b^2)=a^2+b^2.$

261. It follows that the sum of two squares can be factored by the introduction of the imaginary unit. Thus the factors of a^2+b^2 are $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$.

Further, if the denominator of a fraction is of the form $a+b\sqrt{-1}$, it may be rationalized by multiplying the numerator and the denominator by the conjugate expression $a-b\sqrt{-1}$.

262. Complete the work indicated in the following expressions:

1. $(\sqrt{-5})(\sqrt{-6})$.
2. $(\sqrt{-4})(\sqrt{-9})$.
3. $(-\sqrt{8})(\sqrt{-4})$.
4. $(3\sqrt{-3})(2\sqrt{-2})$.
5. $(\sqrt{-a^2})(\sqrt{-a})$.
6. $(\sqrt{-b^2})(\sqrt{-c^2})$.
7. $(-5\sqrt{-x})(3\sqrt{-y})$.
8. $(\sqrt{-2})(\sqrt{-3})(\sqrt{-4})$.
9. $(\sqrt{-2})(\sqrt{-4})(\sqrt{-5})(\sqrt{-8})$.
10. $(2+3\sqrt{-2})(3-2\sqrt{-2})$.
11. $(5-\sqrt{-3})(4+2\sqrt{-5})$.
12. $(2\sqrt{-2}+4\sqrt{-3})(3\sqrt{-3}-5\sqrt{-5})$.
13. $(2+\sqrt{-2})(2-\sqrt{-2})$.

14. $(x - \sqrt{-y})\,(x + \sqrt{-y})$.

15. $(\sqrt{-2} + \sqrt{-3} + \sqrt{-4})\,(\sqrt{-2} - \sqrt{-3} - \sqrt{-4})$.
Ans. $5 + 4\sqrt{3}$.

16. $\{1 - \sqrt{(1 - e^2)}\}\{1 + \sqrt{(1 - e^2)}\}$.

17. $(a\sqrt{-a} + b\sqrt{-b})\,(a\sqrt{-a} - b\sqrt{-b})$.

18. $\left(x - \dfrac{1 + \sqrt{-3}}{2}\right)\left(x - \dfrac{1 - \sqrt{-3}}{2}\right)$.
Ans. $x^2 - x + 1$.

19. $(-\sqrt{18}) \div (\sqrt{-6})$.

20. $(-\sqrt{-15}) \div (-\sqrt{5})$.

21. $(\sqrt{-15}) \div (-\sqrt{5})$.

22. $(\sqrt{15}) \div (\sqrt{-5})$.

23. $(-\sqrt{15}) \div (\sqrt{-5})$.

24. $(-\sqrt{-15}) \div (-\sqrt{-5})$.

25. $(\sqrt{-15}) \div (\sqrt{-5})$.

26. $(6\sqrt{-10}) \div (2\sqrt{-5})$.

27. $8\sqrt{-a^2} \div (2\sqrt{-a})$.

28. $(3 + \sqrt{-2})^2$.

29. $(\sqrt{-3} - 3\sqrt{-1})^2$.

30. $(\sqrt{-1} + 2\sqrt{-2})^3$.

31. $(1 + \sqrt{-3})^3$.

32. $(1 - \sqrt{-3})^3$.

33. $(-\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1})^3$.

34. $(-\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1})^3$.
Ans. 1.

35. $(\sqrt{9 + 40\sqrt{-1}} + \sqrt{9 - 40\sqrt{-1}})^2$.

Express with rational denominator:

36. $\dfrac{1}{3 - \sqrt{2}}$.

37. $\dfrac{4 + \sqrt{-2}}{2 - \sqrt{-2}}$.

38. $\dfrac{a + x\sqrt{-1}}{a - x\sqrt{-1}}$.

39. $\dfrac{(a + \sqrt{-1})^3 - (a - \sqrt{-1})^3}{(a + \sqrt{-1})^2 - (a - \sqrt{-1})^2}$. Ans. $\dfrac{3a^2 - 1}{2a}$.

Find the factors of:

40. $a^2 + 9$.

41. $a + b$.

42. $x^2 - 6x + 13$.
Ans. $(x-3) + 2\sqrt{-1}$, $(x-3) - 2\sqrt{-1}$.

43. $x^2 - 8x + 17$.

44. $x^2 + 3$. Ans. $x + \sqrt{-3}$, $x - \sqrt{-3}$.

45. Find the value of $(-\sqrt{-1})^{4n+3}$, when n is a positive integer.

BINOMIAL SURDS.

263. A **Binomial** in which one or both of the terms are surds is called a *Binomial Surd*.

264. To explain the method of finding the square root of a binomial surd, we square $\sqrt{5} + \sqrt{3}$.

$$(\sqrt{5} + \sqrt{3})^2 = 5 + 2\sqrt{15} + 3 = 8 + 2\sqrt{15}$$

To find the square root of $8 + 2\sqrt{15}$ it is evident, then, that we must find two numbers whose sum is 8 and whose product is 15. The only numbers that answer these conditions are 5 and 3. Hence,

$$\sqrt{8 + 2\sqrt{15}} = \sqrt{5} + \sqrt{3}.$$

1. Find the square root of $11 + 6\sqrt{2}$.

11 is the sum of the two numbers sought, and $6\sqrt{2}$ is *twice* their product, or $3\sqrt{2} = \sqrt{18}$, their product. The only numbers whose sum is 11 and whose product is 18 are 9 and 2. Hence,

$$\sqrt{11 + 6\sqrt{2}} = \sqrt{9} + \sqrt{2} = 3 + \sqrt{2} \quad \text{Ans.}$$

Hence, to find the square root of a binomial surd,

Rule.

Write the binomial surd so that the radical part shall have 2 for its coefficient. Then by inspection find two numbers whose sum is the rational term and whose product is the number under the radical, and connect the square roots of these two numbers with the sign of the radical part.

Find the square root of:

2. $9 + 4\sqrt{5}$.

3. $7 + 4\sqrt{3}$.

4. $27 + 10\sqrt{2}$.

5. $7 - 2\sqrt{10}$.

6. $10 - 6\sqrt{1}$.

7. $23 - 4\sqrt{15}$.

8. $4 + \sqrt{15}$.

$$\sqrt{4 + \sqrt{15}} = \sqrt{\frac{8 + 2\sqrt{15}}{2}} = \frac{\sqrt{5} + \sqrt{3}}{\sqrt{2}} \quad \text{Ans.}$$

9. $\sqrt{32} - \sqrt{30}$.

$$\begin{aligned}\sqrt{(\sqrt{32} - \sqrt{30})} &= \sqrt{\{\sqrt{2}\,(4 - \sqrt{15})\}}\\ &= \sqrt[4]{2}\sqrt{(4 - \sqrt{15})}\\ &= \frac{\sqrt[4]{2}\,(\sqrt{5} - \sqrt{3})}{\sqrt{2}} = \frac{\sqrt{5} - \sqrt{3}}{\sqrt[4]{2}} \quad \text{Ans.}\end{aligned}$$

10. $4 - 2\sqrt{3}$.

11. $9 - 2\sqrt{14}$.

12. $10 + 2\sqrt{21}$.

13. $6 - \sqrt{35}$.

14. $31 + 12\sqrt{-5}$.

15. $-5 + 12\sqrt{-1}$.

16. $\sqrt{2} - \sqrt{-70}$.

17. $a^2 - 1 + 2a\sqrt{-1}$.

18. $4ab - 2(a^2 - b^2)\sqrt{-1}$.

Ans. $(a + b) - (a - b)\sqrt{-1}$.

CHAPTER XVIII.

RADICAL EQUATIONS. PURE EQUATIONS.

265. Radical Equations, that is, equations having the unknown number under the radical sign, require Involution in their reduction.

1. Reduce $4\sqrt{x} - 8 = 8$.

Transposing, $\quad 4\sqrt{x} = 16$
or, $\quad \sqrt{x} = 4$
Squaring, $\quad x = 16$

2. Reduce $\sqrt{x^2 - 3} + x = 3$.

Transposing, $\quad \sqrt{x^2 - 3} = 3 - x$
Squaring, $\quad x^2 - 3 = 9 - 6x + x^2$
or, $\quad 6x = 12$
Ans. $\quad x = 12$

3. Reduce $\sqrt{x + 5} - \sqrt{x - 1} = 2$.

Transposing, $\quad \sqrt{x + 5} = 2 + \sqrt{x - 1}$
Squaring, $\quad x + 5 = 4 + 4\sqrt{x - 1} + x - 1$
Transposing, uniting, and dividing by 2,

$$1 = 2\sqrt{x - 1}$$

Squaring, $\quad 1 = 4x - 4$
Whence, $\quad x = \frac{5}{4}$

Hence, to reduce radical equations, we deduce from these examples the following general

Rule.

Transpose the terms so that a radical part shall stand by itself; then involve each member of the equation to a power of the same degree as the radical; if the unknown number is still under the radical sign, transpose and involve as before; finally reduce as usual.

Reduce the following equations:

4. $7 + \frac{5}{8} + 5\sqrt{x} = \frac{261}{8}$.

5. $\frac{\sqrt{a^2 + \sqrt{x}}}{\sqrt{c}} = \sqrt{c}$.

6. $0.2 + 0.5\sqrt{\frac{20}{x}} = 2.2$.

7. $\frac{\sqrt{x}}{x - c\,x} = \frac{x}{\sqrt{x}}$.

8. $\sqrt{x - 9} = \sqrt{x} - 1$.

9. $(\sqrt{2\,x} + 2)^{\frac{1}{3}} = 2$.

10. $\sqrt{x + 8} = \sqrt{x - 7} + \sqrt{5}$.

11. $\sqrt{x} - \sqrt{x + a} = \frac{a}{\sqrt{x + a}}$. Ans. $-\frac{4}{3}a$.

12. $\sqrt{x^2 - 14} = x - \sqrt{2}$.

13. $\sqrt{x - \sqrt{x - 9} + 17} = \sqrt{x - 23}$.

14. $\frac{\sqrt{x} + 3}{\sqrt{x} - 8} = \frac{\sqrt{x} - 18}{\sqrt{x} + 4}$.

15. $\frac{\sqrt{3\,x} + 8}{\sqrt{3\,x} + 2} = \frac{4\sqrt{3\,x} + 10}{4\sqrt{3\,x} - 2}$. Ans. 3.

16. $\sqrt[3]{5\,x} + 8 = 13$.

17. $\sqrt{8\,x + 17} - \sqrt{2\,x} = \sqrt{2\,x + 9}$. Ans. 8.

18. $\sqrt{x + 3} + \sqrt{x + 8} - \sqrt{4\,x + 21} = 0$.

19. $\sqrt{x + 2} + \sqrt{4\,x + 1} - \sqrt{9\,x + 7} = 0$.

20. $\sqrt{x} + \sqrt{4\,a + x} = 2\sqrt{b + x}$. Ans. $\frac{(a - b)^2}{2\,a - b}$.

21. $\sqrt{x-1}+\sqrt{x}=\dfrac{2}{\sqrt{x}}$.

22. $\sqrt{9+2x}-\sqrt{2x}=\dfrac{5}{\sqrt{9+2x}}$.

23. $\sqrt{x}-\sqrt{x-8}=\dfrac{2}{\sqrt{x-8}}$.

24. $\dfrac{1}{1-x}+\dfrac{1}{\sqrt{x-1}}+\dfrac{1}{\sqrt{x-1}}=0$.

25. $\sqrt{7x-5}+\sqrt{4x-1}=\sqrt{7x-4}+\sqrt{4x-2}$.

(Square without transposing.)

PURE EQUATIONS.

266. A **Pure Equation** is one that contains but *one* power of the unknown number; as,

$$x+ab=c,\ 5x^2+8=13,\ \text{or}\ 7x^m=ac.$$

267. A **Pure Quadratic Equation** is one that contains *only* the second power of the unknown number; as,

$$5x^2-7a=b,\ ay^2+8=d,\ \text{or}\ abx^2=4.$$

268. Equations containing the unknown number involved to any power require Evolution in their reduction.

269. To reduce pure equations containing the unknown number involved to any power.

1. Reduce $\dfrac{3x^2-8}{4}=1$.

$$\frac{3x^2-8}{4}=1$$
$$\frac{3x^2}{4}=3$$
$$x^2=4$$
$$x=\pm 2$$

Performing the division indicated in the first member, transposing, and uniting, we have $\frac{3}{4}x^2=3$; dividing by $\frac{3}{4}$, we have $x^2=4$; extracting the square root of each member of $x^2=4$, we have $x=\pm 2$ (§ 213).

2. Reduce $\frac{x^3}{7} + 50 = 1$.

$$\frac{x^3}{7} + 50 = 1$$
$$\frac{x^3}{7} = -49$$
$$x^3 = 7\,(-49)$$
$$x = -7$$

Transposing and uniting, we have $\frac{x^3}{7} = -49$; clearing of fractions, we have $x^3 = 7(-49)$; extracting the cube root of each member of $x^3 = 7\,(-49)$, we have $x = -7$. Hence,

Rule.

Reduce the equation so as to have as one member the unknown number involved to any degree, and then extract that root of each member which is of the same name as the power of the unknown number.

NOTE. It appears from the solution of Example 1 that *every pure quadratic equation has two roots numerically the same, but with opposite signs.*

Reduce the following equations:

3. $(x-2)^2 = 4 - 4\,x$.

4. $4\,x^2 - 7 = \frac{5}{3}\,x^2 + 14$.

5. $\frac{4}{7}\,x^2 + \frac{5}{7} = \frac{3}{4}\,x^2 - \frac{105}{28}$.

6. $\dfrac{a\,c - x^2}{c} = \dfrac{b\,d - x^2}{d}$.

7. $\dfrac{x-3}{3\,x+1} = \dfrac{3\,x-1}{x+3}$.

270. Equations may require in their reduction both Involution and Evolution; and in this case the rule in Art. 265, as well as that in Art. 269, must be applied. Which rule is first to be applied depends upon whether the expression containing the unknown number is evolved or involved.

8. $\sqrt{x-a} = \dfrac{a-b}{\sqrt{x+a}}$.

Clearing of fractions, $\sqrt{x^2 - a^2} = a - b$

Squaring, $x^2 - a^2 = a^2 - 2\,a\,b + b^2$

Transposing and uniting, $x^2 = 2\,a^2 - 2\,a\,b + b^2$

Extracting the square root, $x = \pm\sqrt{(2\,a^2 - 2\,a\,b + b^2)}$ Ans.

9. $15 + \sqrt{x^3 + 17} = 24.$

10. $\sqrt{\dfrac{5x^2 - 1}{x}} = \sqrt{x}.$

11. $(\sqrt{9 + x^3} - 2)^3 = 64.$ Ans. 3.

12. $\frac{37}{4} - \sqrt[3]{2\left(x^4 + \frac{7}{16}\right)} = \frac{33}{4}.$

271. Simultaneous equations, containing two or more unknown numbers, may require for their reduction Involution, or Evolution, or both. In these equations the elimination is effected by the same principles as in simple equations (§§ 180–182).

13. $\begin{cases} \dfrac{2x^2}{3} + \dfrac{y}{5} = 6.4. \\ 3x^2 - y = 25. \end{cases}$

$$\frac{2x^2}{3} + \frac{y}{5} = 6.4 \qquad (1)$$

$$3x^2 - y = 25 \qquad (2)$$

$$\frac{10x^2}{3} + y = 32$$

$$\frac{19x^2}{3} = 57 \qquad (3)$$

$$x^2 = 9$$

$$x = \pm 3 \qquad (4)$$

$$6 + \frac{y}{5} = 6.4 \qquad (5)$$

$$y = 2 \qquad (6)$$

Adding five times (1) to (2), we obtain (3), which reduced gives (4), or $x = \pm 3$; substituting this value of x in (1), we obtain (5), which reduced gives (6), or $y = 2$.

14. $\begin{cases} x^2yz = 48. \\ xy^2z = 72. \\ xyz^2 = 96. \end{cases}$

15. $\begin{cases} xy = 20. \\ \dfrac{x}{5} + \dfrac{y}{4} = 2(x - y). \end{cases}$ Ans. $\begin{cases} x = 5. \\ y = 4. \end{cases}$

16. $\begin{cases} x^4 - y^4 = 15. \\ x + 2y = 6y - x. \end{cases}$

17. $\begin{cases} 2x + 3y = 10y. \\ x^2 - 4y^2 = 33. \end{cases}$

18. $\begin{cases} 3xy = 24. \\ 4xz = 48. \\ 5yz = 120. \end{cases}$

272. The following problems give equations that are to be reduced according to the principles in Arts. 265–271.

1. What two numbers are as $m : n$, of whose squares the sum is 9?

Let $mx =$ the first number,

then $nx =$ the second number.

Then $m^2x^2 + n^2x^2 = 9$

or,

$$x^2 = \frac{9}{m^2 + n^2}$$

$$x = \pm \frac{3}{\sqrt{m^2 + n^2}}$$

$$\text{Ans.} \begin{cases} mx = \pm \dfrac{3m}{\sqrt{m^2 + n^2}}, \text{ the first number.} \\ nx = \pm \dfrac{3n}{\sqrt{m^2 + n^2}}, \text{ the second number.} \end{cases}$$

2. There is a rectangular field whose area is 6 acres, and whose length is to its breadth as $5 : 3$. Find the length and breadth.

3. The sum of two numbers is 25, and the less divided by the greater is to the greater divided by the less as $4 : 9$. Find the numbers. Ans. 15 and 10.

4. A teacher, being asked how many pupils he had, said, "If you subtract 17 from the number, add 4 to the square root of this remainder, and multiply this sum by 10, I shall have 100." How many had he?

5. A lumber-dealer bought two wood-lots, containing together 210 acres. For each he paid as many dollars an acre as there were acres in the field, and what he paid for the greater was to what he paid for the less as $16 : 9$. Find the area of each. Ans. Greater, 120; less, 90 acres.

6. Find three numbers such that the square of the first multiplied by the second is 45, the square of the second multiplied by the third is 175, and the square of the third multiplied by the first is 147.

CHAPTER XIX.

AFFECTED QUADRATIC EQUATIONS.

273. An **Affected Quadratic Equation** is one that contains only the first and second powers of the unknown number; as,

$$2x^2 - 7x = 15; \text{ or } ax - bx^2 = c.$$

274. Every affected quadratic equation can be reduced to the form

$$x^2 + bx = c,$$

in which b and c represent any numbers whatever, positive or negative, integral or fractional.

For all the terms containing x^2 can be collected into one term whose coefficient we will represent by a; all the terms containing x can be collected into one term whose coefficient we will represent by d; and all the other terms can be united, whose aggregate we will represent by e. Therefore every affected quadratic equation can be reduced to the form

$$ax^2 + dx = e \qquad (1)$$

Dividing (1) by a, $$x^2 + \frac{d}{a}x = \frac{e}{a} \qquad (2)$$

Letting $\frac{d}{a} = b$, and $\frac{e}{a} = c$, we have $$x^2 + bx = c \qquad (3)$$

275. The first member of the equation $x^2 + bx = c$ cannot be a perfect square (§ 222, Note 2). But we know that the square of a binomial is *the square of the first term plus or minus twice the product of the two terms plus the square of the last term*; and if we can find the third term which will make $x^2 + bx$ a perfect square of a binomial, we can then reduce the equation $x^2 + bx = c$.

Since bx has in it as a factor the square root of x^2, x^2 can be the first term of the square of a binomial, and bx the second term of the same square; and since the second term of the square is twice the product of the two terms of the binomial, the last term of the binomial must be the quotient found by dividing the second term of the square by twice the square root of the first term of the square of the binomial; that is, the last term of the binomial is $\frac{bx}{2x} = \frac{b}{2}$; and therefore the third term of the square must be $\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$. Adding $\frac{b^2}{4}$ to each member, we

$$x^2 + bx = c \qquad (1)$$

$$x^2 + bx + \frac{b^2}{4} = \frac{b^2}{4} + c \qquad (2)$$

$$x + \frac{b}{2} = \pm\sqrt{\frac{b^2}{4} + c} \qquad (3)$$

$$x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + c} \qquad (4)$$

have (2), an equation whose first member is a perfect square. Extracting the square root of each member of (2), and transposing, we obtain (4), or $x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + c}$, which is a general expression for the value of x in any equation in the form of $x^2 + bx = c$.

Hence, as every affected quadratic equation can be reduced to the form $x^2 + bx = c$, in which b and c represent any numbers whatever, positive or negative, integral or fractional, every affected quadratic equation can be reduced by the following

Rule.

Reduce the equation to the form $x^2 + bx = c$, *and add to each member the square of half the coefficient of* x.

Extract the square root of each member, and then reduce as in simple equations.

Note 1. Reducing an equation to the form $x^2 + bx = c$ means, not only that all the terms containing x^2 are to be united into one term, and all those containing x into one term, and all the other terms transposed to the right-hand member of the equation, but also that the coefficient of x^2 must be one, and its sign $+$. (See Art. 213, last section of the Rule.)

Reduce the following equations:

1. $5x^2 - 7 = 10x + 68$.

Transposing and uniting, $5x^2 - 10x = 75$

Dividing by 5, $x^2 - 2x = 15$

Completing the square, $x^2 - 2x + 1 = 16$

Evolving, $x - 1 = \pm 4$

Transposing, $x = 1 \pm 4 = 5$, or -3.

NOTE 2. Since, in reducing the general equation $x^2 + bx = c$, we find $x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + c}$, every affected quadratic equation must have two roots; one obtained by considering the expression $\sqrt{\frac{b^2}{4} + c}$ positive, the other by considering this expression negative. Whenever $\sqrt{\frac{b^2}{4} + c} = 0$, these two roots will be equal.

2. $\frac{1}{4} + \frac{x}{6} - \frac{x^2}{6} = \frac{x^2}{12} - \frac{5}{12}$.

Transposing and uniting, $-\frac{x^2}{4} + \frac{x}{6} = -\frac{2}{3}$

Multiplying by -4, $x^2 - \frac{2x}{3} = \frac{8}{3}$

Completing the square, $x^2 - \frac{2x}{3} + \frac{1}{9} = \frac{1}{9} + \frac{8}{3} = \frac{25}{9}$

Evolving, $x - \frac{1}{3} = \pm \frac{5}{3}$

Transposing, $x = \frac{1}{3} \pm \frac{5}{3} = 2$, or $-\frac{4}{3}$

3. $5x^2 - 27 - 10x = 13$. Ans. $x = 4$, or -2.

4. $x - \frac{5}{x} = \frac{2(x + 5)}{x}$. Ans. $x = 5$, or -3.

5. $x - 5 = \frac{x + 3}{x - 6}$. Ans. $x = 9$, or 3.

6. $5x - 23 + \frac{4}{x} = 7 - \frac{41}{x}$. Ans. $x = 3$.

NOTE 3. In this example *both* roots are 3.

7. $3x + \frac{25}{x - 10} = 10$. Ans. $x = 8\frac{1}{3}$, or 5.

8. $\frac{x}{8} - \frac{64 - x}{x - 4} = \frac{x}{4} - 15.$

9. $\frac{10}{7} + \frac{4(x + 5)}{x - 3} = x.$

10. $\frac{x + 8}{x^2 + 1} = \frac{4}{x}.$

11. $5 = \frac{x}{4x^2 - 2} + 2.$

12. $3x - \frac{2x}{x + 3} = \frac{8}{3x + 9}.$

13. $x - \frac{1}{x} = 4 - \frac{x}{2} + \frac{1}{2x}.$

14. $3x^2 - 6ax = 6a + 3.$

Dividing by 3, $x^2 - 2ax = 2a + 1$

Completing the square, $x^2 - 2ax + a^2 = a^2 + 2a + 1$

Evolving, $x - a = \pm(a + 1)$

Whence, $x = a \pm (a + 1)$

$= 2a + 1$, or -1

15. $x^2 - m^2 = 2nx - n^2.$

16. $\frac{1}{m + x + n} = \frac{1}{m} + \frac{1}{x} + \frac{1}{n}.$ Ans. $x = -n$, or $-m$.

17. $x - b = \frac{ab}{x} - a.$

18. $\frac{x + 1}{x^2} = \frac{a + 1}{a^2}.$

19. $(x + 2a)^3 - (x + a)^3 = 37a^3.$

20. $\frac{d}{x - 3c} = \frac{x}{9(d - c)}.$

21. $x^2 + b^2 = 2ax.$

22. $bdx^2 + ac = bcx + adx.$

23. $\frac{x^2 + 1}{x} = \frac{m}{n} + \frac{n}{m}.$

24. $2x + 3a = \frac{2b}{3} + \frac{ab}{x}.$

276. Whenever an equation has been reduced to the form $x^2 + bx = c$, its roots can be written at once; for this equation reduced (§ 275) gives $x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + c}.$ Hence,

The roots of an equation reduced to the form $x^2 + bx = c$ are equal to one half the coefficient of x with the opposite sign, plus or minus the square root of the sum of the square of one half this coefficient and the second member of the equation.

In accordance with this, find the roots of x in the following equations:

25. $x^2 + 10x = 56$.

Ans. $x = -5 \pm \sqrt{25 + 56} = -5 \pm 9 = 4$, or -14.

26. $x^2 - 4x = 12$.

Ans. $x = 2 \pm \sqrt{4 + 12} = 2 \pm 4 = 6$, or -2.

27. $x^2 + 8x = 20$.

28. $x^2 - \dfrac{x}{8} = \dfrac{3}{16}$.

Ans. $x = \frac{1}{16} \pm \sqrt{\frac{1}{256} + \frac{3}{16}} = \frac{1}{16} \pm \frac{7}{16} = \frac{1}{2}$, or $-\frac{3}{8}$.

29. $x^2 + 5x = 50$.

30. $x^2 - 4ax = -4a^2$.

31. $x^2 - (a + b)x = -ab$.

$$\text{Ans. } x = \frac{a+b}{2} \pm \sqrt{\left(\frac{a+b}{2}\right)^2 - ab}$$
$$= \frac{a+b}{2} \pm \sqrt{\left(\frac{a-b}{2}\right)^2}$$
$$= \frac{a+b}{2} \pm \frac{a-b}{2} = a, \text{ or } b$$

32. $x^2 + (m - n)x = mn$.

33. $x^2 - 2(c + d)x = -4cd$.

34. $x^2 = a^2(1 - a)x + a^3$.

35. $7 - 2x = \frac{3}{4}x^2$.

36. $5x^2 - 39 = 2x$.

37. $10x^2 + 30 = 40x$.

38. $cx = ax^2 - b$. Ans. $x = \dfrac{1}{2a}(c \pm \sqrt{c^2 + 4ab})$.

39. $3x^2 - 7x = 13$. Ans. $x = \frac{1}{6}(7 \pm \sqrt{205})$.

40. $\dfrac{7x^2}{5a} - \dfrac{a}{7} = \dfrac{4x}{5}$. Ans. $x = \dfrac{5a}{7}$, or $-\dfrac{a}{7}$.

41. $3x^2 + \dfrac{17x}{2} = 2x^2 - 4$. Ans. $x = -\frac{1}{2}$, or -8.

42. $\dfrac{2x}{14 - x} - \dfrac{x-3}{2} = \dfrac{x-1}{6}$.

43. $\dfrac{2x^2 - 8}{7} - 3x + 7 = \dfrac{x - 3}{3}$.

44. $x^2 - 2bx = (a + b)(a - b)$.

45. $\frac{5}{7}x^2 - x = \frac{24}{7}$.

46. $\dfrac{1}{586} = \dfrac{7}{88x^2 - 30x}$.

47. $\dfrac{2}{x + 2} + \dfrac{9}{5x} = 1$.

48. $x^2 + \frac{2}{7} = \frac{15}{7}x$.

49. $1 - \dfrac{2}{3x} = \dfrac{2}{x + 1}$.

50. $\dfrac{3}{x + 2} = \dfrac{13}{5} - \dfrac{2}{7 - 2x}$.

51. $5 - 2x^{-2} = 3x^{-1}$. (Multiply by x^2.)

52. $2\sqrt{x^5} - 10\sqrt{x} = 3\sqrt{x^3}$. (Divide by $\sqrt{x}$.)

53. $\dfrac{37 - 33x}{x^2 - 6x + 9} = 1$.

54. $1 - 22x^{-2} = \frac{3}{5}x^{-1}$.

55. $\dfrac{x^3 - 2x^2 + 5}{x^2 + 2x - 8} = x - 1$.

56. $(x - 2)(x - 5) = 13\frac{3}{4}$.

57. $\dfrac{x - 2}{x - 1} + \dfrac{x + 1}{x - 3} = \dfrac{2x + 20}{x + 3}$.

NOTE. There are other methods of completing the square, but nothing is gained by their use. The method of finding the roots of an affected quadratic equation given in this article is the shortest method, and it will be used in all the examples that follow.

277. The rule which has been given for the solution of affected quadratic equations applies equally well to any equation containing but two powers of the unknown number *whenever the index of one power is exactly twice that of the other*. By the same reasoning as in Art. 274, it can be shown that all such equations can be reduced to the form

$$a x^{2n} + d x^n = e,$$

or

$$x^{2n} + b x^n = c.$$

It will be seen that the first member is composed of two terms so related that they may be the first two terms of a binomial square, and we can supply the third by the rule already given for completing the square.

Reduce the following equations:

1. $x^4 - 4x^2 = 45.$

$$x^4 - 4x^2 = 45 \quad (1)$$
$$x^4 - 4x^2 + 4 = 4 + 45 = 49 \quad (2)$$
$$x^2 - 2 = \pm 7 \quad (3)$$
$$x^2 = 9, \text{ or } -5 \quad (4)$$
$$x = \pm 3, \text{ or } \pm\sqrt{-5} \quad (5)$$

Since the square root of x^4 is x^2, it is evident that the second term contains as one of its factors the square root of the first term; that is, the first member of the equation is composed of two terms so related that they may be the first two terms of the square of a binomial. Completing the square, we have (2); extracting the square root of each member of (2), we obtain (3); transposing, we have (4); and extracting the square root of (4), we have $x = \pm 3$, or $\pm\sqrt{-5}$.

Or, reducing the equation by the rule in Art. 276,

$$x^4 - 4x^2 = 45$$
$$x^2 = 2 \pm \sqrt{4 + 45} = 2 \pm 7 = 9, \text{ or } -5$$
$$x = \pm 3, \text{ or } \pm\sqrt{-5}$$

2. $5x^{\frac{3}{2}} + 3x^{\frac{3}{4}} = 344.$

$$5x^{\frac{3}{2}} + 3x^{\frac{3}{4}} = 344$$
$$x^{\frac{3}{2}} + \tfrac{3}{5}x^{\frac{3}{4}} = \tfrac{344}{5}$$
$$x^{\frac{3}{4}} = -0.3 \pm \sqrt{0.09 + 68.8} = -0.3 \pm 8.3 = 8, \text{ or } -8.6$$
$$x = 16, \text{ or } (-8.6)^{\frac{4}{3}}$$

3. $x^4 + \dfrac{x^2}{2} = \dfrac{3}{16}.$ Ans. $x = \pm\frac{1}{2}$, or $\pm\frac{1}{2}\sqrt{-3}$.

4. $x^6 - 2x^3 = 48.$ Ans. $x = 2$, or $-\sqrt[3]{6}$.

5. $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} = 2.$ Ans. $x = 8$, or $-\frac{1}{8}$.

6. $x^6 - \frac{3}{4}x^3 = 58$. Ans. $x = 2$, or $-\sqrt[3]{7\frac{1}{4}}$.

7. $\dfrac{\sqrt{x}+6}{\sqrt{x}} = \dfrac{2\sqrt{x}+6}{\sqrt{x}+1}$. Verify both answers.

8. $x - \frac{3}{5}\sqrt{x} = 22$.

9. $\frac{3}{5}x^{2n} - \frac{1}{4}x^n = 57\frac{1}{2}$.

10. $x^6 - a\,x^3 = b^2$.

11. $x^{-4} + 4\,x^{-2} = 32$.

278. A polynomial may take the place of the unknown number in an affected quadratic equation. In this case the equation can be reduced by considering the polynomial as a single term.

Reduce the following equations:

1. $(x+5)^2 - 4(x+5) = 32$.

$$(x+5)^2 - 4(x+5) = 32 \quad (1)$$
$$(x+5)^2 - 4(x+5) + 4 = 36 \quad (2)$$
$$x + 5 - 2 = \pm 6 \quad (3)$$
$$x = 3, \text{ or } -9 \quad (4)$$

Considering $x+5$ as a single term and completing the square, we have (2); extracting the square root, transposing, &c., we have (4), or $x = 3$, or -9.

Or, by Art. 276, from (1), we have at once

$$x + 5 = 2 \pm \sqrt{4+32} = 2 \pm 6 = 8, \text{ or } -4$$
$$x = 3, \text{ or } -9$$

Note. We might put $(x+5) = y$; then $(x+5)^2 = y^2$, and the equation becomes $y^2 - 4y = 32$. After finding the value of y in this equation, $x+5$ must be substituted for y.

2. $(2x+3)^2 - (2x+3) = 42$. Ans. $x = 2$, or $-4\frac{1}{2}$.

3. $\sqrt{19-x} - \sqrt[4]{19-x} = 2$. Verify both answers.

4. $x^2 - x + 4 + \sqrt{x^2 - x + 4} = 2$.

Ans. $x = \frac{1}{2}(1 \pm \sqrt{-11})$, or 1, or 0.

Verify these answers.

5. $(2x-3)^2+2\sqrt{2x-3}=\dfrac{3}{2x-3}$.

Ans. $x=2$, or $\frac{1}{2}(3+\sqrt[3]{9})$.

6. $x^2+2x-\sqrt{x^2+2x-6}=12$.

(Subtract 6 from both sides.)

7. $4x+7+\sqrt{4+2x-x^2}=2x^2$.

8. $\dfrac{3+x^2}{2}=x+\sqrt{6-2x+x^2}$.

9. $1+\frac{1}{5}\sqrt{x^2+x+5}=\dfrac{10}{\sqrt{x^2+x+5}}$.

MISCELLANEOUS EXAMPLES.

279. In the following examples, when the answers are not given, the answers obtained should be proved by substituting them in the original equation.

Reduce the following equations:

1. $x+\dfrac{4}{x}+5=0$. Ans. $x=-1$, or -4.

2. $\dfrac{1}{2x}+\dfrac{2}{3x^2}=\dfrac{7}{6}$. Ans. $x=1$, or $-\frac{4}{7}$.

3. $\dfrac{x^2}{2}-\dfrac{25}{6}=\dfrac{x}{9}$. Ans. $x=3$, or $-2\frac{7}{9}$.

4. $(x+3)(x-3)=6x-14$. Ans. $x=5$, or 1.

5. $(x+2)^3+61=(x+3)^3$.

6. $5x^2+8=2x^2+8x+24$.

7. $(2x+3)^2=(x+8)^2$.

8. $\dfrac{9-x}{2}-\dfrac{x+3}{5}=\dfrac{6-x}{2x-13}$.

9. $(x+2)^2-(2x+1)^2=(x+1)^2$.

10. $\dfrac{x+4}{x-4}-\dfrac{x-4}{x+4}=4\frac{4}{5}$.

11. $(10x+1)^2=9(16x+17)$. Ans. $x=2$, or -0.76.

12. $8x-\dfrac{4x-1}{x-2}=5x+\dfrac{3x^2+9}{x+1}$.

13. $\dfrac{3x+2}{x+1}=\dfrac{4x+6}{2x+5}$.

14. $\dfrac{x-3a}{b}=\dfrac{9(b-a)}{x}$.

15. $\dfrac{x+c}{x-c}-\dfrac{x-c}{x+c}=1$. Ans. $x=(2\pm\sqrt{3})\,c$.

16. $\dfrac{4x}{x+10}-\dfrac{5-2x}{2x+4}=\dfrac{21}{8}$.

20. $\dfrac{2+x}{2-x}-\dfrac{9}{5}=\dfrac{1-x}{1+x}$.
Ans. $x=\frac{2}{3}$, or -3.

17. $\dfrac{x+3}{x-2}-\dfrac{4}{3}=\dfrac{3x+1}{2x+2}$.

18. $\dfrac{24}{1-x}-\dfrac{8}{x}=12$.

21. $\dfrac{1}{2+x}+\dfrac{3}{10}=\dfrac{1}{2x-4}$.

19. $\dfrac{120}{x}-\dfrac{120}{x+1}=\dfrac{36}{x+2}$.

22. $\dfrac{3}{x}+\dfrac{1}{5}=\dfrac{2x+1}{x^2}$.

23. $\dfrac{3x-2}{2x-3}-\dfrac{2x-3}{3x-2}=\dfrac{15}{4}$.

24. $\dfrac{x^2}{3}-\frac{3}{4}x=3(x+1)$.

25. $(x-1)(x-2)-6(x-3)(x-4)=0$.

26. $5x-\dfrac{15-x}{x+4}=15$.

27. $x^{-1}+x^{-2}=6$.

28. $(x+2)(x+3)=3x(x-2)+2(2x+1)$.

29. $\dfrac{45}{5-x} - 28 + \dfrac{6x}{7} = 47 - \dfrac{9x}{7}$.

30. $5\sqrt{x} - 3\sqrt{x-9} = 13$. Ans. $x = 25$, or $9\frac{49}{64}$.

31. $4\sqrt{16+x} = 7\sqrt{16+x} - 2x + 3$.

32. $3\sqrt{x-a} + 2\sqrt{10x} = \dfrac{17a + 7x}{\sqrt{x-a}}$.

Ans. $x = 10a$, or $-\frac{5}{3}a$.

33. $\sqrt{3x+1} + \dfrac{35}{\sqrt{3x+1}} = 3\sqrt{x}$.

34. $\sqrt{2x-1} - \sqrt{x-1} = 1$.

35. $\sqrt{3x+4} + \sqrt{2x+1} = \sqrt{11x+5}$.

36. $x - 1 = \sqrt{1 + \sqrt{5x^3 - 4x^2}}$.

37. $\frac{1}{4}\sqrt[5]{x^2} + 2\sqrt[5]{x} - \frac{13}{16} = \frac{23}{16}$.

38. $5 - 4x^{-1} + 7x^{-2} = 19x^{-2}$. Ans. $x = 2$, or -1.2.

39. $\sqrt{2x+1} - \sqrt[4]{2x+1} = 6$.

40. $\dfrac{x^4}{4} - \dfrac{2}{3} - \dfrac{3x^2}{7} = \dfrac{25}{3} + \dfrac{53x^2}{7} - \dfrac{3x^4}{4}$.

Ans. $x = \pm 3$, or $\pm\sqrt{-1}$.

41. $4x^6 - \frac{3}{16} = \frac{1}{8} - 2x^3$. Ans. $x = \frac{1}{2}$, or $\frac{1}{2}\sqrt[3]{-5}$.

42. $3\sqrt{x} - \frac{4}{3} = \frac{35}{3} - \frac{1}{2}\sqrt[4]{x}$.

43. $\dfrac{x}{b^2c(n+x)} = \dfrac{n+x}{a^2cx}$. Ans. $x = \dfrac{\pm bn}{a \mp b}$.

44. $x^2 - a^2 = \dfrac{1}{a^2} - \dfrac{1}{x^2}$.

45. $\dfrac{4b-x}{a^2-ab}+\dfrac{x-2a+2b}{ax}=\dfrac{7b-4a}{ax-bx}.$

Ans. $x=2a+b$, or $-a+2b$.

46. $\dfrac{x}{a^2-4b^2}-\dfrac{x-5b}{(a+2b)x}=\dfrac{1}{a+2b}+\dfrac{x-2a+19b}{(a-2b)x}.$

Ans. $x=2a+6b$, or $a-8b$.

47. $\dfrac{18a}{5a-x-3b}-1=\dfrac{x+a}{x+2b}.$

Ans. $x=\dfrac{a}{2}-3b$, or $-5a-b$.

48. $x^4-5(a^2+b^2)x=-(2a^2+3ab-2b^2)^2.$

Ans. $x=\pm(2a-b)$, or $\pm(a+2b)$.

49. $x^2+x=\dfrac{2c+cx}{a-b}+2.$

Ans. $x=\dfrac{a-b+c}{a-b}$, or -2.

50. $x+\dfrac{1}{x}=1+\dfrac{1}{1-\dfrac{4b^2}{a^2+3b^2}}.$ Ans. $x=\dfrac{a\pm b}{a\mp b}.$

51. $\dfrac{b^2+3}{3b^2+1}=\dfrac{x^2+x+1}{x^2-x+1}.$ Ans. $x=\dfrac{1\pm b}{1\mp b}.$

52. $x^2-\sqrt{3x^2-2x+3}=\dfrac{2x-5}{3}.$

Ans. $x=1$, or $-\frac{1}{3}$, or $\frac{1}{3}(1\pm\sqrt{-5})$.

53. $(x+4)(x-1)=4\sqrt{x(x+3)-2}-2.$

Ans. $x=3$, or -4, or $\frac{1}{2}(-3\pm\sqrt{17})$.

54. $2\sqrt{x^2-3x+4}=5x^2-15x+4.$

Ans. $x=3$, or 0, or 2.4, or 0.6.

55. $abx^2+\dfrac{3a^2x}{c}=\dfrac{6a^2+ab-2b^2}{c^2}-\dfrac{b^2x}{c}.$

56. $\dfrac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}-\dfrac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=8\sqrt{x^2-1}.$

Ans. $x=\pm 2$, or ± 1.

57. $x^{\frac{1}{n}}-6x^{\frac{1}{2n}}+9=0.$

58. $\dfrac{x}{ab(a+b)}\left(\dfrac{x}{a+b}-1\right)=\dfrac{1}{(a-b)^2}.$

Ans. $x=\dfrac{a(a+b)}{a-b}$, or $\dfrac{b(a+b)}{b-a}$.

59. $\dfrac{1}{(a^2-1)^2}-\dfrac{1}{x^2}\left(\dfrac{1}{(a+1)^4}-\dfrac{1}{x^2(a^2-1)^2}+\dfrac{1}{(a-1)^4}\right)=0.$

Ans. $x=\pm\dfrac{a\pm 1}{a\mp 1}$.

60. $$\frac{x^2}{(a^2+b^2)(4a-3b)^2}-\frac{6x}{(5a^2+b^2)(a^2+5b^2)(4a-3b)}$$
$$=-\frac{1}{(5a^2+b^2)(a^2+5b^2)(a^2+b^2)}.$$

PROBLEMS

PRODUCING AFFECTED QUADRATIC EQUATIONS WITH BUT ONE UNKNOWN NUMBER.

280. The negative values obtained in solving the following Problems always satisfy the equations formed in accordance with the given conditions, and can generally be explained, and the words of the Problem so changed as to reverse the signs (§§ 189, 190). In all cases where it is possible, the student should be required to explain the negative answers, and change the problem so as to change the negative to positive answers.

1. A man who had travelled 400 miles found that if he had travelled 10 miles less each day he would have taken 2 days more to complete his journey. How many days was he on the road, and what was his daily rate?

Let $\qquad x =$ number of days on the road;

then $\qquad \frac{400}{x} =$ his daily rate.

Then $$\frac{400}{x} - \frac{400}{x+2} = 10$$

or, $$\frac{40}{x} - \frac{40}{x+2} = 1$$

Clearing of fractions,

$$40x + 80 - 40x = x^2 + 2x$$

or, $$x^2 + 2x = 80$$

Whence, $\qquad x = -1 \pm 9 = 8$, or -10, number of days,

and $\qquad \frac{400}{x} = 50$, or -40, daily rate.

The negative answers verify in the equation. In the problem, if *more* and *less* change places, the answers 10 days and 40 miles a day will be found correct.

2. Divide 60 into two parts, such that the sum of their squares shall be 2138. Ans. 43 and 17.

Notice that, in solving this example, if we consider the square root plus, we get the greater part; if minus, the less.

3. A merchant who sold a piece of cloth for \$72 found that, if the price which he paid for it were multiplied by his gain, the product would be equal to the cube of the gain. Find the gain. Ans. \$8.

If the word "gain" were changed to *loss*, the other answer, -9, or as it would then become $+9$, would be correct.

4. The sum of the squares of two consecutive numbers is 145. What are the numbers?

5. A gentleman distributed among some boys \$10. If he had begun by giving each 5 cents more, 10 of the boys would have received nothing. How many boys were there, and how much did he give to each?

6. A man who had travelled 224 miles found that, in order to return the same road in 1 day less, he must travel 4 miles more each day. Find how many days he had travelled, and how many miles a day.

7. Find two numbers whose difference is 6, and the sum of whose fourth powers is 2402.

Let $x - 3$ and $x + 3$ be the numbers.

8. A man hires a piece of land for \$330. He lets all but 25 acres for \$5 an acre more than he pays for it, receiving enough to pay for the whole. How many acres does he hire?

9. A cistern is filled by two pipes in 2 h. 55 m. By the greater alone it can be filled in 2 hours less time than by the smaller alone. Find the time for each pipe.

10. Find two numbers whose sum is 9, and the sum of whose cubes is 189.

11. A drover sold a cow for \$56, and found that if the price which he paid for it were multiplied by his gain, the product would be equal to the cube of the gain. What was his gain? Ans. \$7.

12. A farmer, having built 42 rods of fence, found that, if he had built one more rod a day, he would have completed the work in one day less time. How many rods did he build a day, and how many days did he work?

13. The length of a rectangular field exceeds the breadth 1 rod, and the area is $3\frac{3}{4}$ acres. Find the dimensions.

14. Find a number such that 4 times its square less 6 times the number itself is 270.

15. A man sold a horse for \$275, gaining a per cent expressed by a twenty-fifth of the cost of the horse. How much did the horse cost him? Ans. \$250.

16. From a cask containing 50 gallons of white syrup enough was drawn to fill a small keg, and the same quantity of darker colored syrup was put in; then the same keg was filled again from the cask, and then there were 32 gallons of the original white syrup left in the cask. How many gallons did the keg hold?

17. A merchant bought a piece of cloth for \$54. He used for himself 8 yards, and sold the rest for 20 cents more a yard than he gave, receiving for what he sold \$51.80. How many yards did he buy? Ans. 45 yards.

18. There is a rectangular piece of land 84 rods long and 52 rods wide, and just within the boundaries is a ditch of uniform breadth running entirely round the land. Within the ditch the area is 26 acres and 73 square rods. Find the width of the ditch.

19. The sum of the two figures of a certain number of two figures is 11, and their product is 19 less than the number expressed by the figures in reverse order. What is the number?

20. What are eggs a dozen, when 4 more in 40 cents' worth lowers the price 4 cents a dozen? Ans. 24 cents.

21. Two numbers are as 4 : 3, and their product plus their sum is 62. Find the numbers.

22. Three students, A, B, and C, began on the same day to solve a number of problems. A solved 10 a day, and finished them 4 days before B. B solved 2 more a day than C, and finished them 5 days before C. How many problems were there, and how many days did each work?

23. A broker sells some railroad shares for \$3150. He afterward buys for the same sum 7 more shares at \$5 less a share. Find the number of shares he sold, and the price.

Ans. 63 shares at \$50 a share.

24. Two horsemen start at the same time from two places 18 miles apart. At the end of 12 hours the second horseman overtakes the first, and on comparing their rates they find that there has been a difference in their rates of 2 minutes in each mile. Find their rates and the distance each has travelled.

25. A rectangular piece of land has an area of 3 acres, and if its length is decreased 5 rods, and its breadth increased 4 rods, its area is increased 20 square rods. Find the dimensions.

26. A man bought a number of $100 railway shares, at a certain rate per cent discount a share, for $8100, and afterwards sold for $8250 all but 15 at the same rate per cent premium a share. How many shares did he buy, and what did he give a share?

27. A man walks at a uniform rate on a road which passes over a bridge distant 18 miles from the point which the man has reached at noon. If his rate were half a mile an hour less than it is, the time at which he crosses the bridge would be 1 h. 12 m. later than it is. Find his rate, and the time at which he crosses the bridge.

Ans. Rate, 3 miles an hour; time, 6 P. M.

28. A man bought a number of sheep for $400; he sold all but 20 for $352, gaining $0.40 a head. How many sheep did he buy, and at what price?

29. Several boys on an excursion spent each the same amount of money. If there had been 6 more boys in the party, and each had spent 25 cents less, the sum spent would have been $39. If there had been 4 less, and each had spent 25 cents more, the sum spent would have been $32. Find the number of boys, and the amount each spent.

Ans. 20 boys; amount spent by each, $1.75.

CHAPTER XX.

SIMULTANEOUS QUADRATIC EQUATIONS

CONTAINING TWO UNKNOWN NUMBERS.

281. The **Degree** of an equation is shown by the sum of the indices of the unknown numbers in that term in which this sum is the greatest. Thus,

$7x - 3xy = 9$	is an	equation	of the	second	degree,
$ax^2y + x^3y = bc^3$	"	"	"	fourth	"
$8x - 7y^4 = x^3y^2$	"	"	"	fifth	"

NOTE. Before deciding of what degree an equation is, if the unknown numbers appear both in the denominators and in the numerators or integral terms, it must be cleared of fractions, and also from negative and fractional exponents.

282. A **Homogeneous Equation** is one in which the sum of the exponents of the unknown numbers in each term containing unknown numbers is the same. Thus,

$$x^2 + 2xy + y^2 = 25$$

or $$x^3 - 3xy^2 + 3x^2y - y^3 = 64$$

or $$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 625$$

is a homogeneous equation.

283. Two numbers enter **Symmetrically** into an equation when, whatever their values, they can exchange places without destroying the equation. Thus,

$$x^2 - 2xy + y^2 = 36$$

or
$$x^3 + 3x^2y + 3xy^2 + y^3 = 64$$

or
$$x^2 - 2xy + y^2 + 5x + 5y = 100$$

284. Simultaneous quadratic equations containing two unknown numbers can generally be solved by the rules already given, if they come under one of the three following cases:

I. When one of the equations is simple and the other quadratic.

II. When the unknown numbers enter symmetrically into each equation.

III. When each equation is quadratic and homogeneous.

CASE I.

285. When one of the Equations is Simple and the other Quadratic.

1. Solve $\begin{cases} x + y = 5. \\ x^2 - 2y^2 = 1. \end{cases}$

$x + y = 5 \quad (1) \qquad\qquad x^2 - 2y^2 = 1 \quad (2)$

$y = 5 - x \quad (3) \qquad x^2 - 50 + 20x - 2x^2 = 1$

$x^2 - 20x = -51 \quad (4)$

$y = -12, \text{ or } 2 \quad (6) \qquad\qquad x = 17, \text{ or } 3 \quad (5)$

From (1) we obtain (3), or $y = 5 - x$. Substituting this value of y in (2), we obtain (4), an affected quadratic equation, which reduced gives (5); and substituting these values of x in (3), we obtain (6).

In this Case the values of the unknown numbers can generally be found by *substituting in the quadratic equation the value of one unknown number found by reducing the simple equation.*

2. Solve $\begin{cases} xy = 20. \\ x + y = 9. \end{cases}$

$$xy = 20 \qquad (1)$$

$$\begin{aligned} x + y &= 9 && (2) \\ x^2 + 2xy + y^2 &= 81 && (3) \\ 4xy &= 80 && (4) \\ \hline x^2 - 2xy + y^2 &= 1 && (5) \\ x - y &= \pm 1 && (6) \\ \hline 2x &= 10, \text{ or } 8 && (7) \\ 2y &= 8, \text{ or } 10 && (8) \\ x &= 5, \text{ or } 4 && (9) \\ y &= 4, \text{ or } 5 && (10) \end{aligned}$$

Subtracting four times (1) from the square of (2), we obtain (5); extracting the square root of each member of (5), we obtain (6); adding (6) to (2), we obtain (7); subtracting (6) from (2), we obtain (8); and reducing (7) and (8), we obtain (9) and (10).

NOTE. Though Example 2 can be solved by the same method as Example 1, the method given is preferable.

Solve the following equations:

3. $\begin{cases} xy = 15. \\ x - y = 2. \end{cases}$

4. $\begin{cases} x - y = 3. \\ x^2 - y^2 = 33. \end{cases}$

5. $\begin{cases} x + y = 13. \\ x^2 - y^2 = 39. \end{cases}$

6. $\begin{cases} x - y = 2. \\ x^2 + y^2 = 164. \end{cases}$

7. $\begin{cases} xy = 24. \\ 5x - y = 37. \end{cases}$

8. $\begin{cases} xy = 5. \\ 3x - 2y = 13. \end{cases}$

9. $\begin{cases} \dfrac{x}{3} + \dfrac{y}{4} = 5. \\ \dfrac{3}{x} + \dfrac{4}{y} = \dfrac{5}{6}. \end{cases}$

10. $\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = 2.9. \\ 2x - 3y = 4. \end{cases}$

11. $\begin{cases} x + y = 8. \\ x^2 + y^2 = 50. \end{cases}$

Case II.

286. When the Unknown Numbers enter symmetrically into each Equation.

1. Solve $\begin{cases} x + y = 9. \\ x^3 + y^3 = 243. \end{cases}$

$$\begin{array}{rlr} x + y = & 9 & (1) \\ x^2 + 2xy + y^2 = & 81 & (3) \\ x^2 - xy + y^2 = & 27 & (4) \\ \hline 3xy = & 54 & (5) \\ xy = & 18 & (6) \\ \hline x^2 - 2xy + y^2 = & 9 & (7) \\ x - y = & \pm 3 & (8) \\ \hline 2x = & 12, \text{ or } 6 & (9) \\ 2y = & 6, \text{ or } 12 & (10) \\ x = & 6, \text{ or } 3 & (11) \\ y = & 3, \text{ or } 6 & (12) \end{array} \qquad x^3 + y^3 = 243 \quad (2)$$

Squaring (1), we obtain (3); dividing (2) by (1), we obtain (4); subtracting (4) from (3), we obtain (5), from which we obtain (6); subtracting (6) from (4), we obtain (7); extracting the square root of each member of (7), we obtain (8); adding (8) to (1), we obtain (9); subtracting (8) from (1), we obtain (10); and reducing (9) and (10), we obtain (11) and (12).

Note 1. It must not be inferred that x and y are equal to each other in these equations; for when $x = 6$, $y = 3$; and when $x = 3$, $y = 6$. In all the equations under this Case the values of the two unknown numbers are interchangeable.

Note 2. Although $x^3 + y^3 = 243$ is not a quadratic equation, yet, as we can combine the two given equations in such a manner as to produce at once a quadratic equation, we introduce it here.

2. Solve $\begin{cases} xy = 6. \\ x^2 + y^2 - 2x - 2y = 3. \end{cases}$

$$xy = 6 \quad (1) \qquad \begin{aligned} x^2 + y^2 - 2x - 2y &= 3 \\ 2xy \qquad\qquad &= 12 \end{aligned} \qquad (2)$$

$$(x+y)^2 - 2(x+y) = 15 \qquad (3)$$

$$x + y = 1 \pm 4 = 5, \text{ or } -3 \qquad (4)$$

$$x = 3, \text{ or } 2, \text{ or } \frac{-3 \pm \sqrt{-15}}{2} \qquad (5)$$

$$y = 2, \text{ or } 3, \text{ or } \frac{-3 \mp \sqrt{-15}}{2} \qquad (6)$$

Adding twice (1) to (2), we obtain (3); from (3) we obtain (4); and combining (4) and (1) as the sum and product are combined in Ex. 2, Art. 285, we obtain (5) and (6).

In Case II. the process varies as the given equations vary. In general the equations are reduced by *a proper combination of the sum of the squares, or the square of the sum or of the difference, with multiples of the product of the two unknown numbers; and finally, of the sum, with the difference of the two unknown numbers.*

NOTE 3. When the unknown numbers enter into each equation symmetrically in all respects except their signs, the equations can be reduced by this same method; e. g. $x - y = 7$, and $x^3 - y^3 = 511$. In such equations the values of the unknown numbers are not interchangeable.

NOTE 4. The signs $\pm$ $\mp$ standing before any number taken independently are equivalent to each other; but when one of two numbers is equal to $\pm a$ while the other is equal to $\mp b$, the meaning is that the first is equal to $+a$, when the second is equal to $-b$; and the first to $-a$, when the second is equal to $+b$.

By this method solve the following equations:

3. $\begin{cases} 5x - 5y = 35. \\ 2x^3 - 2y^3 = 1022. \end{cases}$ Ans. $\begin{cases} x = 8, \text{ or } -1. \\ y = 1, \text{ or } -8. \end{cases}$

4. $\begin{cases} x + y = 10. \\ x^3 + y^3 = 370. \end{cases}$

5. $\begin{cases} x - y = 9. \\ x^2 + y^2 = 45. \end{cases}$

6. $\begin{cases} x^3 + y^3 = 91. \\ x^2 - xy + y^2 = 13. \end{cases}$

7. $\begin{cases} x - y = 4. \\ x^3 - y^3 = 316. \end{cases}$ Ans. $\begin{cases} x = 7, \text{ or } -3. \\ y = 3, \text{ or } -7. \end{cases}$

8. $\begin{cases} x^3 - y^3 = 875. \\ x^2 + xy + y^2 = 175. \end{cases}$

9. $\begin{cases} x + \sqrt{xy} + y = 7. \\ x^2 + xy + y^2 = 21. \end{cases}$

10. $\begin{cases} x - \sqrt{xy} + y = 13. \\ x^2 + xy + y^2 = 481. \end{cases}$

CASE III.

287. When each Equation is Quadratic and Homogeneous.

1. Solve $\begin{cases} x^2 - 2xy = 15. \\ 3xy - y^2 = 14. \end{cases}$

$$x^2 - 2xy = 15 \quad (1) \qquad 3xy - y^2 = 14 \quad (2)$$

Let $x = vy$

$$v^2y^2 - 2vy^2 = 15 \quad (3) \qquad 3vy^2 - y^2 = 14 \quad (4)$$

$$y^2 = \frac{15}{v^2 - 2v} \quad (5) \qquad y^2 = \frac{14}{3v - 1} \quad (6)$$

$$\frac{15}{v^2 - 2v} = \frac{14}{3v - 1} \quad (7)$$

$$45v - 15 = 14v^2 - 28v \quad (8)$$

$$v^2 - \tfrac{73}{14}v = -\tfrac{15}{14} \quad (9)$$

$$v = 5, \text{ or } \tfrac{3}{14} \quad (10)$$

$$y^2 = \frac{14}{15 - 1}, \text{ or } \frac{14}{\frac{9}{14} - 1} \quad (11)$$

$$y = \pm 1, \text{ or } \pm \tfrac{14}{5}\sqrt{-5} \quad (12)$$

$$x = vy = \pm 5, \text{ or } \pm \tfrac{3}{5}\sqrt{-5} \quad (13)$$

Substituting vy for x in (1) and (2), we obtain (3) and (4); from (3) and (4) we obtain (5) and (6); putting these two values of y^2 equal to each other, we obtain (7), which reduced gives (10); substituting this value of v in (6), we obtain (11), which reduced gives (12); and substituting in $x = vy$ the values of v and y from (10) and (12), we obtain (13).

Examples under Case III. can generally be reduced best by *substituting for one of the unknown numbers the product of the other by some unknown number, and then finding the value of this third unknown number.* When the value of this third number becomes known, the values of the given unknown numbers can be readily found by substitution.

By this method solve the following equations:

2. $\begin{cases} 3xy + y^2 = 7. \\ 2x^2 - xy = 6. \end{cases}$ Ans. $\begin{cases} x = \pm 2, \text{ or } \pm 0.3\sqrt{10}. \\ y = \pm 1, \text{ or } \mp 1.4\sqrt{10}. \end{cases}$

3. $\begin{cases} x^2 + 2xy = 21. \\ x^2 + y^2 = 13. \end{cases}$ 4. $\begin{cases} x^2 - xy = 10. \\ 5xy - 3y^2 = 48. \end{cases}$

5. $\begin{cases} x^2 - 3xy = 2y^2 - 68. \\ xy - 5y^2 = x^2 - 92. \end{cases}$ Ans. $\begin{cases} x = \pm 6, \text{ or } \pm \frac{26}{83}\sqrt{83}. \\ y = \pm 4, \text{ or } \pm \frac{40}{83}\sqrt{83}. \end{cases}$

6. $\begin{cases} 2x^2 - 1 = y^2 + 3xy. \\ 8 - x^2 = y^2 + 3. \end{cases}$

MISCELLANEOUS EXAMPLES.

288. Solve the following equations:

NOTE. Some of the examples given below belong at the same time to two Cases. Thus in Example 1 both the equations are symmetrical, and both are quadratic and homogeneous, and therefore it belongs both to Case II. and Case III. Example 2 belongs both to Case I. and Case II.

1. $\begin{cases} xy = 18. \\ x^2 + y^2 = 45. \end{cases}$ 2. $\begin{cases} x + y = 18. \\ x^2 + y^2 = 164. \end{cases}$

3. $\begin{cases} x^2 + xy = 126. \\ xy - y^2 = 20. \end{cases}$ Ans. $\begin{cases} x = \pm 7\sqrt{2}, \text{ or } \pm 9. \\ y = \pm 2\sqrt{2}, \text{ or } \pm 5. \end{cases}$

4. $\begin{cases} xy = 24. \\ x^2 + y^2 = x - y + 50. \end{cases}$ 5. $\begin{cases} 3x^2 - 5y^2 = -5. \\ x^2 + 3xy = 69 + y^2. \end{cases}$

6. $\begin{cases} \dfrac{x^3 - y^3}{x - y} = 7. \\ 5xy + 3 = 13. \end{cases}$ 7. $\begin{cases} x + y = 5. \\ x^2y^2 - 4xy = 12. \end{cases}$

(In Ex. 7, considering xy a single number, find its value in the second equation.)

8. $\begin{cases} x^2y - xy^2 = 20. \\ x^3 - y^3 = 61. \end{cases}$

(Subtract from the second equation three times the first, and extract the cube root of each member of the resulting equation.)

9. $\begin{cases} x^2y + xy^2 = 330. \\ x^3 + y^3 = 341. \end{cases}$ Ans. $\begin{cases} x = 6, \text{ or } 5. \\ y = 5, \text{ or } 6. \end{cases}$

10. $\begin{cases} xy = 14. \\ 27(x^3 + y^3) = 13(x + y)^3. \end{cases}$

11. $\begin{cases} x + y = 12. \\ x^3 + y^3 = 18xy. \end{cases}$ 13. $\begin{cases} xy = 2. \\ x^4 + y^4 = 17. \end{cases}$

12. $\begin{cases} 4x^2y + 4xy^2 = 80. \\ 3x^3y^2 + 3x^2y^3 = 240. \end{cases}$ 14. $\begin{cases} xy = 12. \\ x^4 - y^4 = 175. \end{cases}$

15. $\begin{cases} 3xy = 45. \\ x^2 + 5y = 24 + 5x - y^2. \end{cases}$

Ans. $\begin{cases} x = \frac{1}{2}(3 \pm \sqrt{69}), \text{ or } 5, \text{ or } -3. \\ y = -\frac{1}{2}(3 \mp \sqrt{69}), \text{ or } 3, \text{ or } -5. \end{cases}$

16. $\begin{cases} x^{\frac{2}{3}} y^{\frac{3}{2}} = 9y^2. \\ 4x^{\frac{1}{3}} + 3y^{\frac{1}{2}} = 15. \end{cases}$ 19. $\begin{cases} x^{\frac{1}{3}} + y^{\frac{1}{3}} = 5. \\ x + y = 35. \end{cases}$

17. $\begin{cases} 5x^2 + 3xy = 270. \\ 2y^2 - xy = 20. \end{cases}$ 20. $\begin{cases} x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1. \\ x - y = 37. \end{cases}$

18. $\begin{cases} x + y = 25. \\ \sqrt{x} + \sqrt{y} = x - y. \end{cases}$ 21. $\begin{cases} x^{-1} + y^{-1} = \frac{5}{6}. \\ x^{-2} + y^{-2} = \frac{13}{36}. \end{cases}$

22. $\begin{cases} x^3 - 2x^2y + 2xy^2 - y^3 = 7. \\ x^3 - x^2y + xy^2 - y^3 = 13. \end{cases}$

Ans. $\begin{cases} x = 3, \text{ or } -2. \\ y = 2, \text{ or } -3. \end{cases}$

23. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 1\frac{1}{5}. \\ \dfrac{1}{x^3} + \dfrac{1}{y^3} = 1\frac{1}{125}. \end{cases}$

28. $\begin{cases} x^{-1} - y^{-1} = \frac{1}{12}. \\ x^{-3} - y^{-3} = \frac{37}{1728}. \end{cases}$

24. $\begin{cases} x + y = 4. \\ x^4 + y^4 = 82. \end{cases}$

29. $\begin{cases} x - y = 1. \\ x^4 + y^4 = 337. \end{cases}$

25. $\begin{cases} x^3 - y^3 = 208. \\ x^2 + xy + y^2 = 52. \end{cases}$

30. $\begin{cases} x - y = a. \\ x^2 + y^2 = 5a^2. \end{cases}$

26. $\begin{cases} x - y = 1. \\ x^5 - y^5 = 211. \end{cases}$

31. $\begin{cases} x - y = m + n. \\ 4xy = 15m + 10mn. \end{cases}$

27. $\begin{cases} x + y = 6. \\ x^5 + y^5 = 3126. \end{cases}$

32. $\begin{cases} x + y = 5m. \\ x^3 + y^3 = 35m^3. \end{cases}$

33. $\begin{cases} x + y = 10. \\ \dfrac{x^4}{y^2} + \dfrac{y^4}{x^2} = \dfrac{1225}{9} - 2xy. \end{cases}$

Ans. $\begin{cases} x = 6, \text{ or } 4, \text{ or } 5 \pm \frac{5}{11}\sqrt{-143}. \\ y = 4, \text{ or } 6, \text{ or } 5 \mp \frac{5}{11}\sqrt{-143}. \end{cases}$

PROBLEMS

PRODUCING SIMULTANEOUS QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN NUMBERS.

289. Some of the following problems may be solved with one unknown number. In the answers given, the negative values are omitted.

1. There is a certain number of two figures, whose product is 10; and if 56 is added to the number, and the sum of the squares of its figures subtracted from the sum, the order of the figures will be reversed. Find the number.

2. The area of a rectangular field is 975 square rods, and if the length were decreased 5 rods, and the breadth increased 5 rods, the area would be 1020 square rods. What are the length and breadth?

Ans. Length, 39 rods; breadth, 25 rods.

3. The difference of the cubes of two numbers is 316, and the sum of their squares plus their product is 79. What are the numbers?

4. The area of a rectangular field whose perimeter is 88 rods is 363 square rods. Find the length and breadth.

5. The fore wheels of a carriage make 8 more revolutions than the hind wheels in going 160 yards, but if the circumference of each wheel is increased 3 feet the carriage must pass over 240 yards in order that the fore wheels may make 8 revolutions more than the hind wheels. What is the circumference of the wheels?

Ans. Fore wheels, 12 feet; hind wheels, 15 feet.

6. There are two pieces of cloth of different lengths. The difference of the squares of the number of yards in each is 76; and the product of the numbers representing the number of yards in each plus one half the square of the number of yards in the longer is 560. Find the length of each.

7. Find two numbers such that 4 times the square of the greater minus 5 times the square of the less shall be 20, and 5 times the square of the less plus their product shall be 100.

8. A drover bought 10 oxen and 18 cows for \$1040, buying one more ox for \$150 than cows for \$60. Find the price of an ox, and the price of a cow. Ans. Ox, \$50; cow, \$30.

9. Find two numbers such that the sum, the product, and the sum of their squares shall be equal to one another.

10. A and B, talking of their ages, find that the square of A's age minus the product of the ages of both is 385, and

3 times this product plus the square of B's age is 3096. Find the age of each.

11. A and B purchased a wood-lot containing 100 acres, each agreeing to pay \$5000. Before paying for the lot, A offered to pay \$10 an acre more than B, if B would allow A to have his choice in the division of the lot. How many acres should each receive, and at what price an acre?

Ans. $\begin{cases} \text{A, 47.5 acres, at \$105.25– an acre.} \\ \text{B, 52.5 acres, at \$ 95.25+ an acre.} \end{cases}$

12. Two sums of money, amounting to \$9000, are loaned at such a rate of interest that the income from each is the same. But if the first part had been at the same rate as the second, the income from it would have been \$160; while if the second had been at the same rate as the first, the income from it would have been \$250. Find the rate of each.

Ans. 1st, 5%; 2d, 4%.

13. A boat's crew, rowing at half their usual rate, row 4 miles down a river and back in 3 hours and 20 minutes. At their usual rate they can go over the same course in 1 hour and 20 minutes. Find the usual rate of the crew, and the rate of the current.

14. A, working alone, built 8 rods of wall; then he hired B to work with him, and at the end of 3 days from the time A began they had completed 24 rods. Again, A worked 3 days and B 1 day, building 21.6 rods. Find the number of rods each can build a day, and the number of days B worked.

15. A reservoir is filled in 10 hours by means of several pipes, through which the water flows at a uniform rate. If there were 2 less pipes, and each pipe discharged 50 gallons more an hour, the reservoir would be filled in 12 hours and 30 minutes. If there were 3 more pipes, and each pipe discharged 25 gallons less an hour, the reservoir would be filled in 7 hours and 30 minutes. Find the number of pipes, and the capacity of the reservoir.

16. A tailor bought two pieces of cloth for \$100. For the first he paid $\frac{1}{5}$ as many dollars a yard as there were yards in both; and for the second $\frac{1}{2}$ as many dollars a yard as there were yards in the first more than in the second; and the first piece cost 3 times as much as the second. Find the number of yards, and the cost a yard of each.

17. A man walks 3 hours at the rate of 3 miles an hour, and then takes a different rate. After a number of hours he finds that, if he had kept his first rate, he would have been 2 miles less distant from his starting point, while if he had walked at his first rate 2 hours, and at his second rate 5 hours, he would be half a mile farther from his starting point. Find the whole time of the walk, and the entire distance travelled.

Ans. 7 hours, and 23 miles.

18. The soldiers of a regiment can be arranged so as to have twice as many men in a line as there are lines. But 16 men must be added to the regiment in order that the men may be arranged in a hollow square six deep, having the same number of men in each outer side of the square as there were in the lines before. Find the number of men in the regiment.

19. The area of a certain rectangle is equal to the area of a square whose side is 6 meters less than one of the sides of the rectangle, and also equal to the area of another rectangle whose length is 2 meters less, and width 1 meter more. Find the length of the sides of the rectangle.

20. At 7 o'clock A.M., A and B set out in opposite directions on their bicycles, from the same point. A's hourly rate was 5 miles. B, riding at a fixed rate, after a while turned and followed A. Three hours after he turned, B passed the point where A was when B turned, and at 12 o'clock he had reduced the distance between them at the time of turning one half. Find B's rate, the time when he turned, the distance between A and B at that time, and the time when B will overtake A if both continue at the same rate of speed.

CHAPTER XXI.

PROPERTIES OF QUADRATIC EQUATIONS.

290. EVERY affected quadratic equation can be reduced (§ 274) to the form

$$x^2 + bx + c = 0.$$

This reduced (§ 275) gives

$$x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}.$$

Thus, there are two roots of a quadratic equation, and the sum of these two roots, $-\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}$ and $-\frac{b}{2} - \sqrt{\frac{b^2}{4} - c}$, is $-b$; and their product is c.

That is, the *sum* of the roots is equal to the coefficient of the second term with its sign changed; the *product* of the roots is equal to the last term.

291. A quadratic equation, $x^2 + bx + c = 0$, will have two real and unequal roots, two real but equal roots, or two imaginary roots, according as b^2 is $>$, $=$, or $< 4c$.

For the roots are $$\frac{-b + \sqrt{b^2 - 4c}}{2},$$

and $$\frac{-b - \sqrt{b^2 - 4c}}{2}.$$

Now if $b^2 > 4c$, the expression under the radical is positive, and the roots are real and unequal.

If $b^2 = 4c$, the roots are real and equal.

If $b^2 < 4c$, the expression under the radical is negative, and both roots are imaginary (§ 213).

292. To illustrate this subject still further, we solve the following example.

1. $x^2 + 7x + 12 = 0$.

Transposing, $\qquad x^2 + 7x = -12$

Whence $\qquad x = -\frac{7}{2} \pm \sqrt{\frac{49}{4} - 12} = -\frac{7}{2} \pm \frac{1}{2} = -3, \text{ or } -4$

Now, since $\qquad x = -3$

and also $\qquad x = -4$

then $\qquad x + 3 = 0 \qquad (1)$

and $\qquad x + 4 = 0 \qquad (2)$

Multiplying (1) by (2), $\qquad x^2 + 7x + 12 = 0 \qquad (3)$

Now (3) is the equation given in the example.

In general, if $\qquad x = a \text{ or } b$

then $\qquad x - a = 0 \qquad (1)$

and $\qquad x - b = 0 \qquad (2)$

Multiplying (1) by (2), $\qquad x^2 - (a + b)x + ab = 0 \qquad (3)$

Equation $\qquad x^2 - (a + b)x + ab = 0$

reduced, gives $\qquad x = a, \text{ or } b$

Thus, it will be seen that a quadratic equation having any two given roots can be formed by *subtracting each of the roots from x, multiplying these two expressions together, and making the product equal to zero.*

2. Write an equation whose roots are -3 and 5.
Ans. $x^2 - 2x - 15 = 0$.

Form the equations whose roots are:

3. $-1, \quad 3$.	8. $5, \ -3$.	13. $6, \ -\frac{1}{3}$.
4. $2, \ -5$.	9. $4, \ -1$.	14. $5, \ -2\frac{2}{5}$.
5. $-3, \ -1$.	10. $\frac{1}{2}, \ -\frac{1}{3}$.	15. ± 4.
6. $-2, \quad 4$.	11. $2, \ \sqrt{-3}$.	16. $\pm(2 + \sqrt{3})$.
7. $1, \ -1$.	12. $6, \ -4$.	17. $\pm(a - b)$.

293. If $m\,n = 0$, either $m = 0$, or $n = 0$. If we know that m is not equal to 0, then we know that n is equal to 0; and if we know that n is not equal to 0, then we know that m is equal to 0. So, if $l\,m\,n = 0$, at least one of the factors must be equal to 0; and so on, for any number of factors.

In the following examples find the roots:

1. $x^2 + 3\,x - 10 = 0$.

Or, $(x - 2)(x + 5) = 0$
Then $x - 2 = 0$, or $x + 5 = 0$
that is, $x = 2$, or -5

2. $x^2 - x - 20 = 0$.

3. $2\,x^2 + 16\,x + 24 = 0$.

4. $x^3 - 27\,x^2 + 50\,x = 0$.

Or, $x(x - 2)(x - 25) = 0$
Whence $x = 0$, or $x - 2 = 0$, or $x - 25 = 0$
Then $x = 0$, or 2, or 25

5. $x^3 - 14\,x^2 - 51\,x = 0$.

6. $x^3 - 1 = 0$.

Factoring, $(x - 1)(x^2 + x + 1) = 0$
Then $x - 1 = 0$ (1), or $x^2 + x + 1 = 0$ (2)
From (1), $x = 1$
From (2), $x^2 + x = -1$
Whence $x = -\frac{1}{2} \pm \sqrt{\frac{3}{4}} = -\frac{1}{2}(1 \pm \sqrt{3})$

7. $x^3 = 8$.

Or, $x^3 - 8 = 0$
Factoring, $(x - 2)(x^2 + 2\,x + 4) = 0$
Then $x - 2 = 0$ (1), or $x^2 + 2\,x + 4 = 0$
From (1), $x = 2$
From (2), $x^2 + 2\,x = -4$
Whence $x = -1 \pm \sqrt{-3}$

8. $x^3 - 7\,x^2 + 10\,x = 0$.

9. $x^3 = 27$.

294. From these examples it will be seen that, if the first member of an equation whose second member is 0 can be divided by the unknown number plus any number, the negative of this second number is one of the roots of the equation.

10. $x^4 - 7x^2 - 6x = 0.$

Or, $x(x+1)(x+2)(x-3) = 0$
Then (§ 293), $x = 0$, or -1, or -2, or 3

NOTE. The difficulty in this example is in finding the factors. That x is a factor, is apparent. Then the part $x^3 - 7x - 6$ can be written

$$x^3 + 2x^2 - 2x^2 - 4x - 3x - 6 = x^2(x+2) - 2x(x+2) - 3(x+2)$$
$$= (x+2)(x^2 - 2x - 3)$$

295. From these examples it might be inferred that the number of roots in an equation containing only one unknown number is the same as the index of the highest power of the unknown number in the equation. But it can be proved that, in an equation containing only one unknown number, the number of different roots cannot exceed the index of the highest power of the unknown number. Thus,

Let $x - a = 0$, a form to which all equations containing only the first power of an unknown number can be reduced.

Then $x = a$

Now suppose b and c are the roots of this equation. Substituting for x, b and c, we have

$$b = a \qquad (1)$$
$$c = a \qquad (2)$$

Therefore, $b = c$; that is, the equation $x - a = 0$ does not have two different roots, that is, it has only one root.

Again:

Let $x^2 + mx + n = 0$
Suppose it has three different roots, a, b, and c.

Then, substituting these values for x, we have

$$a^2 + am + n = 0 \quad (1)$$
$$b^2 + bm + n = 0 \quad (2)$$
$$c^2 + cm + n = 0 \quad (3)$$

Subtracting (2) from (1),

$$a^2 - b^2 + (a - b)m = 0$$

or $$(a - b)(a + b + m) = 0 \quad (4)$$

Also, subtracting (3) from (1),

$$a^2 - c^2 + (a - c)m = 0$$

or $$(a - c)(a + c + m) = 0 \quad (5)$$

Now, since a, b, and c are different roots, neither $a - b$ nor $a - c$ can equal 0.

Therefore, $$a + b + m = 0 \quad (6)$$

and $$a + c + m = 0 \quad (7)$$

Subtracting (7) from (6), $$b - c = 0$$

or $$b = c$$

That is, b and c are not different roots; that is, there are only two different roots.

Again:

Let $$x^3 + mx^2 + nx + p = 0$$

Suppose this equation has four *different* roots, a, b, c, and d. Then, substituting these values for x, we have

$$a^3 + a^2m + an + p = 0 \quad (1)$$
$$b^3 + b^2m + bn + p = 0 \quad (2)$$
$$c^3 + c^2m + cn + p = 0 \quad (3)$$
$$d^3 + d^2m + dn + p = 0 \quad (4)$$

Subtracting successively (2), (3), and (4) from (1),

$$a^3 - b^3 + (a^2 - b^2)m + (a - b)n = 0 \quad (5)$$
$$a^3 - c^3 + (a^2 - c^2)m + (a - c)n = 0 \quad (6)$$
$$a^3 - d^3 + (a^2 - d^2)m + (a - d)n = 0 \quad (7)$$

These factored become

$$(a - b)\{a^2 + ab + b^2 + (a + b)m + n\} = 0 \quad (8)$$
$$(a - c)\{a^2 + ac + c^2 + (a + c)m + n\} = 0 \quad (9)$$
$$(a - d)\{a^2 + ad + d^2 + (a + d)m + n\} = 0 \quad (10)$$

Now, since a is not equal to b, or c, or d, $a-b$, $a-c$, and $a-d$ are not 0. Therefore, from (8), (9), and (10), we have

$$a^2 + ab + b^2 + am + bm + n = 0 \qquad (11)$$
$$a^2 + ac + c^2 + am + cm + n = 0 \qquad (12)$$
$$a^2 + ad + d^2 + am + dm + n = 0 \qquad (13)$$

Subtracting (12) and (13) from (11), we have

$$ab - ac + b^2 - c^2 + (b-c)m = 0 \qquad (14)$$
$$ab - ad + b^2 - d^2 + (b-d)m = 0 \qquad (15)$$

These factored become

$$(b-c)(a+b+c+m) = 0 \qquad (16)$$
$$(b-d)(a+b+d+m) = 0 \qquad (17)$$

As before, $b-c$ and $b-d$ are not 0; therefore, from (16) and (17), we have

$$a+b+c+m = 0 \qquad (18)$$
$$a+b+d+m = 0 \qquad (19)$$

Subtracting (19) from (18), we have

$$c - d = 0 \qquad (20)$$

or

$$c = d$$

That is, c and d are not different roots; that is, there are only three different roots.

In like manner, for higher powers it can be shown that the number of different roots cannot exceed the index of the highest power of the unknown number in the equation.

It must not be inferred, however, that an equation cannot have two, or more, *equal* roots. The equation $x^2 - 2ax + a^2 = 0$, that is, $(x-a)(x-a) = 0$, has two roots, both of which are a.

It follows, too, that if any expression containing only one unknown number is multiplied by this unknown number minus any number, and if this last number is substituted for the unknown number in the product, this product will prove to be 0. For example, multiplying $x^2 - 5x + \frac{1}{2}$ by $x - 2$, we have $x^3 - 7x^2 + \frac{21}{2}x - 1$. Substituting 2

for x, we have $8 - 28 + 21 - 1 = 0$. Or, again, if a is a root of any equation containing only one unknown number, as x, then, if the numbers are transposed so as to make the second member 0, the first member is divisible by $x - a$.

296. This principle gives another method of factoring an algebraic expression. For if, putting the expression equal to 0, we can then reduce the equation thus formed so as to obtain its roots, the expression formed by subtracting any one of these roots from the number whose value we have found will be a factor of the given expression. For example,

Find the factors of:

1. $13x^2 + 221x - 234 = 0$.

Put $$13x^2 + 221x - 234 = 0$$
Then $$x^2 + 17x = 18$$
Whence $$x = \frac{17}{2} \pm \sqrt{\frac{289}{4} + 18} = -\frac{17}{2} \pm \frac{19}{2} = 1, \text{ or } -18$$

Then the factors of $13x^2 + 221x - 234$ are 13, $x - 1$, and $x + 18$.

2. $6x^2 - 102x - 504 = 0$.

3. $2x^2 - 5x + 3 = 0$.

4. $x^3 - 2x^2 - 5x + 6 = 0$.

By trial we find that 1 will satisfy this equation, therefore 1 is one of its three roots.

Dividing $x^3 - 2x^2 - 5x + 6 = 0$ by $x - 1$
gives $$x^2 - x - 6 = 0$$
Or, $$x^2 - x = 6$$
Whence $$x = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 6} = \frac{1}{2} \pm \frac{5}{2} = 3, \text{ or } -2$$

Hence the three roots are 1, -2, and 3. Therefore the required factors are $x-1$, $x-3$, and $x+2$.

When from a cubic equation one of the roots is thus removed, the equation becomes quadratic, and the other two roots can be found as previously shown.

If the roots are integral, it is clear that they are among the prime factors of the last term of the first member.

5. $x^3 + 5x^2 - 2x - 24 = 0.$

6. $x^3 - 6x^2 - x + 30 = 0.$

7. $x^3 + 6x^2 - 37x + 30 = 0.$

8. $8x^3 - 56x^2 - 520x + 1848 = 0.$

9. $9x^3 - 9x^2 - 2x + 2 = 0.$

10. $3x^2 + 9x - 54 = 0.$ 11. $11x^2 + 77x - 198 = 0.$

12. $x^8 + 13x^4 - 68 = 0.$

$$x^4 = 4, \text{ or } -17$$
$$x^2 = \pm 2, \text{ or } \pm\sqrt{-17}$$
$$x = \pm\sqrt{\pm 2}, \text{ or } \pm\sqrt{\pm\sqrt{-17}}.$$

13. $x^6 + 15ax^3 + 44a^2 = 0.$

$$x = -\sqrt[3]{4a}, \text{ or } -\sqrt[3]{11a}.$$

14. $x^4 - 18bx^2 + 65b^2 = 0.$

297. From what has gone before it is evident that, if we can find one root of any equation, another equation can be derived from it in which the highest power of the unknown number will be one less. By various artifices of this nature, equations involving higher powers of the unknown number can be reduced.

1. Find the three cube roots of 64.

2. Find the four roots in $x^4 - 1 = 0.$

3. Find the six roots in $x^6 - 1 = 0.$

298. *If a rational and integral polynomial in terms of x is divided by $x - a$ until a remainder independent of x is obtained, this remainder is the value of the polynomial when $x = a$.*

If U is the polynomial, Q the quotient, and R the remainder,

$$U = Q(x - a) + R.$$

Now this equation is true for all values of x. If, then, $x = a$, $U = R$, since R is independent of x.

For example,

$$\frac{2x^3 + 3x - 8}{x - 2} = 2x^2 + 4x + 11 + \frac{14}{x - 2}$$

$$\therefore \quad 2x^3 + 3x - 8 = 14, \text{ if } x = 2$$

$$\text{or,} \quad 2x^3 + 3x - 22 = 0, \text{ if } x = 2$$

If, in $2x^3 + 3x - 8$, we substitute 2 for x, we have $2 \cdot 2^3 + 3 \cdot 2 - 8 = 14$; and 14 is the remainder if we divide $2x^3 + 3x - 8$ by $x - 2$.

The proposition in this article includes the particular case, viz., that if any rational and integral polynomial in terms of x vanishes when we substitute a for x, then $x - a$ is a factor of the expression. (See Art. 295, last paragraph.)

EXAMPLES.

1. Find the remainder when $3x^3 - 5x^2 + 3x - 4$ is divided by $x - 3$.

2. Find the remainder when $2x^4 + 3x^2 - 4x + 12$ is divided by $x + 4$.

3. Prove that $x + 3$ is a factor of $2x^4 + 5x^3 - 3x - 56$.

4. Prove that $x - 1$ is a factor of any rational and integral polynomial in terms of x, if the sum of the coefficients (including the coefficient of x^0) is zero.

CHAPTER XXII.

RATIO, PROPORTION, AND VARIATION.

299. **Ratio** is the relation of one number to another; or it is the quotient obtained by dividing one number by another.

Ratio is indicated by writing the two numbers one after the other with two dots between, or by expressing the division in the form of a fraction. Thus, the ratio of a to b is written, $a : b$, or $\frac{a}{b}$; read, a is to b, or a divided by b.

300. The **Terms** of a ratio are the numbers compared, whether simple or compound.

The first term of a ratio is called the *antecedent*, the other the *consequent;* the two terms together are called a *couplet.*

301. An **Inverse** or **Reciprocal Ratio** of any two numbers is the ratio of their *reciprocals.* Thus, the *direct* ratio of a to b is $a : b$, that is, $\frac{a}{b}$; the *inverse* ratio of a to b is $\frac{1}{a} : \frac{1}{b}$, that is, $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, or $b : a$.

302. Two numbers are *commensurable* if there is a third number of the same kind which is contained an exact number of times in each. This third number is called the *common measure* of these two numbers. Thus, m and n are commensurable if there is a third number, d, that is contained an exact number of times in each; as, for exam-

ple, 0.7 times in m, and 0.5 times in n; and d is the common measure of m and n. Then $m = 0.7\,d$, and $n = 0.5\,d$, and $m : n = 0.7\,d : 0.5\,d = 7 : 5$, or $\frac{m}{n} = \frac{7}{5}$.

Two numbers are *incommensurable* if they have no common measure.

The ratio of two numbers, as m and n, whether commensurable or not, is expressed by $\frac{m}{n}$. If m and n are incommensurable, $\frac{m}{n}$ is called an *incommensurable ratio.*

A *constant ratio* is a ratio which remains the same, though its terms may vary. Thus, the ratio of $3:4$, $6:8$, $9:12$, is constant; also the ratio of $A : B$ and $mA : mB$.

303. Proportion is an equality of ratios. Four numbers are in proportion when the ratio of the first to the second is equal to the ratio of the third to the fourth.

The equality of two ratios is indicated by the sign of equality ($=$), or by four dots ($::$).

Thus, $a : b = c : d$, or $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$; read, a to b equals c to d, or a is to b as c is to d, or a divided by b equals c divided by d.

In a proportion the antecedents and consequents of the two ratios are respectively the *antecedents* and *consequents* of the proportion. The first and fourth terms are called the *extremes*, and the second and third the *means.*

304. A **Continued Proportion** is a series of equal ratios; as, $a : b = c : d = e : f = g : h$, or $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$.

305. When three numbers are in proportion, for example, $a : b = b : c$, the second is called a *mean proportional* between the other two; and the third, a *third proportional* to the first and second.

306. A proportion is transformed by **Alternation** when antecedent is compared with antecedent, and consequent with consequent.

307. A proportion is transformed by **Inversion** when the antecedents are made consequents, and the consequents antecedents.

308. A proportion is transformed by **Composition** when in each couplet the sum of the antecedent and consequent is compared with the antecedent or with the consequent.

309. A proportion is transformed by **Division** when in each couplet the difference of the antecedent and consequent is compared with the antecedent or with the consequent.

THEOREMS.

310. *In a proportion the product of the extremes is equal to the product of the means.*

Let $$a : b = c : d$$

that is, $$\frac{a}{b} = \frac{c}{d}$$

Clearing of fractions, $$a\,d = b\,c$$

311. *If the product of two numbers is equal to the product of two others, the factors of either product may be made the extremes, and the factors of the other the means, of a proportion.*

Let $$a\,d = b\,c$$

Dividing by $b\,d$, $$\frac{a}{b} = \frac{c}{d}$$

that is, $$a : b = c : d$$

312. *If four numbers are in proportion, they will be in proportion by alternation, and by inversion.*

Let $a : b = c : d$

By Art. 310, $ad = bc$

By Art. 311, $a : c = b : d$, alternation.

or, $b : a = d : c$, inversion.

313. *If four numbers are in proportion, they will be in proportion by composition and by division.*

Let $$a : b = c : d$$

that is, $$\frac{a}{b} = \frac{c}{d}$$

Adding ± 1 to each member, $$\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$$

or, $$\frac{a \pm b}{b} = \frac{c \pm d}{d}$$

that is, $a + b : b = c + d : d$, composition.

or, $a - b : b = c - d : d$, division.

314. *Corollary.* Since, by Art. 313, if $a : b = c : d$

$$a + b : b = c + d : d$$

and also, $$a - b : b = c - d : d$$

by Art. 312, and Art. 36, Ax. 8,

$$a + b : a - b = c + d : c - d$$

315. *Equimultiples of two numbers have the same ratio as the numbers themselves.*

For $$\frac{a}{b} = \frac{ma}{mb}$$

that is, $$a : b = ma : mb$$

316. *Corollary.* It follows that either couplet of a proportion may be multiplied or divided by any number, and the resulting numbers will be in proportion. And since, by Art. 312, if $a : b = ma : mb$, $a : ma = b : mb$, or $ma : a = mb : b$, it follows that both consequents, or both antecedents, may be multiplied or divided by any number, and the resulting numbers will be in proportion.

317. *If four numbers are in proportion, like powers or like roots of these numbers will be in proportion.*

Let $$a : b = c : d$$

that is, $$\frac{a}{b} = \frac{c}{d}$$

Hence, $$\frac{a^n}{b^n} = \frac{c^n}{d^n}$$

that is, $$a^n : b^n = c^n : d^n$$

Since n may be either integral or fractional, the theorem is proved.

318. *In a continued proportion any antecedent is to its consequent as the sum of any number of the antecedents is to the sum of the corresponding consequents.*

Let $$a : b = c : d = e : f$$

Now, $$a\,b = a\,b \qquad (1)$$

and by Art. 310, $$a\,d = b\,c \qquad (2)$$

and also, $$a\,f = b\,e \qquad (3)$$

Adding (1), (2), (3), $$a\,(b + d + f) = b\,(a + c + e)$$

Hence, by Art. 312, $$a : b = a + c + e : b + d + f$$

319. *If there are two sets of numbers in proportion, their products, or quotients, term by term, will be in proportion.*

Let $$a : b = c : d$$

and $$e : f = g : h$$

By Art. 310, $$a\,d = b\,c \qquad (1)$$

and $$e\,h = f\,g \qquad (2)$$

Multiplying (1) by (2), $$a\,d\,e\,h = b\,c\,f\,g \qquad (3)$$

Dividing (1) by (2), $$\frac{a\,d}{e\,h} = \frac{b\,c}{f\,g} \qquad (4)$$

From (3) by Art. 311, $$a\,e : b\,f = c\,g : d\,h$$

and from (4), $$\frac{a}{e} : \frac{b}{f} = \frac{c}{g} : \frac{d}{h}$$

320. The proofs which have been given for commensurable numbers are also true for incommensurable numbers. For the ratio of two incommensurable numbers can be expressed to any required degree of accuracy.

Suppose, for example, it is required to find the ratio of two incommensurable numbers, a and b, to a degree of accuracy within $\frac{1}{100}$. Let the less, b, be divided into 100 equal parts, and suppose a contains 217 such parts with a remainder less than one of the parts, then we have

$$\frac{a}{b} = \frac{217}{100} \text{ within } \frac{1}{100},$$

that is, $\frac{100}{217}$ is the approximate ratio of a to b to the required degree of accuracy.

Or, to make the reasoning general, let b be divided into n equal parts, and suppose a contains m such parts with a remainder less than one of the parts, then we have

$$\frac{a}{b} = \frac{m}{n} \text{ within } \frac{1}{n}.$$

As n may be taken as great as we please, $\frac{1}{n}$ may be made as small as we please, and $\frac{m}{n}$ will be the ratio of a to b to any required degree of accuracy.

A good example of the ratio of incommensurable numbers is the ratio of $\sqrt{2}$ to 1. The $\sqrt{2}$ is a surd (§ 241), but its value can be found to any required degree of accuracy. Thus, $\sqrt{2} > 1.4142$ but < 1.4143; hence $\frac{\sqrt{2}}{1} = 1.4142$ within 0.0001.

321. *Two incommensurable ratios are equal, if their approximate numerical values are always equal when both ratios are expressed to the same degree of accuracy.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ be two incommensurable ratios, whose approximate numerical values are always the same when expressed to the same degree of accuracy; then

$$\frac{a}{b}=\frac{c}{d}.$$

Let the numerical ratio $\frac{m}{n}=\frac{a}{b}$ accurate within $\frac{1}{n}$;

then, by hypothesis, $\frac{m}{n}=\frac{c}{d}$ accurate within $\frac{1}{n}$.

That is, $\frac{a}{b}$ and $\frac{c}{d}$ differ by a quantity less than $\frac{1}{n}$. But as n may be taken as great as we please, $\frac{1}{n}$ may be made as small as we please. Now, if it is possible for the ratios to differ at all, $\frac{1}{n}$ can be made less than that difference, unless that difference is actually zero; that is, they do not differ,

and $$\frac{a}{b}=\frac{c}{d}.$$

322. The laws that have been discussed in Ratio and Proportion of *numbers* apply also to *quantities.* But it must be understood that only *quantities of the same kind* can have a ratio one to another, and the ratio itself, that is, the quotient of one *quantity* divided by another of the same kind is an *abstract number.* The two ratios that form a proportion need not be of the same kind. Thus, if A receives \$70 for 35 days' work, and \$14 for 7 days' work, we can say,

$$\$70 : \$14 = 35 \text{ days} : 7 \text{ days}.$$

In this case the ratio of \$70 to \$14 is 5, an abstract number, and equal to the ratio of 35 days to 7 days.

323. A good example of two incommensurable *quantities* is the ratio of the diagonal to the side of a square. If the *numbers*, s and d, represent the measurement of the side and diagonal, respectively, then

$$\frac{d}{s} = \frac{\sqrt{2}}{1} = \sqrt{2};$$

and this, as in Art. 320, can be found to any required degree of accuracy.

324. EXERCISES IN PROPORTION.

1. If $\dfrac{a^2+b^2}{ab+bc} = \dfrac{ab+bc}{b^2+c^2}$, prove that $a:b = b:c$.

Clearing the given equation of fractions,

$$a^2b^2 + a^2c^2 + b^4 + b^2c^2 = a^2b^2 + 2ab^2c + b^2c^2$$
$$a^2c^2 - 2ab^2c + b^4 = 0$$
$$ac - b^2 = 0$$
$$ac = b^2$$

Or (§ 311), $a : b = b : c$

2. If $\dfrac{a}{b} = \dfrac{c}{d}$, prove that $\dfrac{a}{c} = \sqrt{\dfrac{a^2+b^2}{c^2+d^2}}$.

3. If $\dfrac{a^2-b^2}{a^2} = \dfrac{c^2-d^2}{c^2}$, prove that $\dfrac{a}{b} = \dfrac{c}{d}$.

4. If $\dfrac{a+b+c+d}{a+b-c-d} = \dfrac{a-b+c-d}{a-b-c+d}$, prove that $\dfrac{a}{b} = \dfrac{c}{d}$.

5. If $10a+b : 10c+d = 8a+b : 8c+d$, prove that $a : b = c : d$.

6. If $a : b = c : d = e : f$, prove that $ma - nc : mb - nd = e : f$.

7. Prove that, if the first term of a proportion is greatest, the last term is least.

8. If $a+b : c+d = c-d : a-b$, and $a+b$ is the greatest, prove that $b > d$.

Find the value of x in

9. $\dfrac{2b}{c}\sqrt[12]{\dfrac{a^8 b^{16}}{c^3}} : x = \dfrac{2(3ac)^2}{\sqrt[15]{b^9 c^5}} : \dfrac{3\sqrt{a}}{4b}$.

10. $\dfrac{6}{b} : \dfrac{1}{3a\sqrt{a}}\left(\dfrac{a}{\sqrt[3]{b}}\right)^2 = x : \sqrt{\left(\dfrac{b}{\sqrt[6]{16a}}\right)^3}$.

11. $\left(\dfrac{10\sqrt[3]{a^2}}{3\sqrt[4]{b^5}}\right)^2 : x = \sqrt{\dfrac{5a\sqrt[3]{a^2}}{4\sqrt[5]{a}\cdot b^9}} : \dfrac{9b^{-3}}{\sqrt{5}}$.

Solve the following equations:

12. $\begin{cases} 3x^3 y = 81. & (1) \\ x^3 + y^3 : x^3 - y^3 = 14 : 13. & (2) \end{cases}$

From (2), by Art. 315, $2x^3 : 2y^3 = 27 : 1$
By Art. 317, $x^3 : y^3 = 27 : 1$
By Art. 318, $x : y = 3 : 1$
By Art. 310, $x = 3y$ (3)
From (1) and (3) we find $x = 3$ and $y = 1$.

13. $\begin{cases} x - y : y = 200 : x. & (1) \\ x - y : x = \quad 8 : y & (2) \end{cases}$

From (1) and (2), by Art. 320, $(x-y)^2 : xy = 1600 : xy$
By Arts. 317 and 318, $x - y : 1 = 40 : 1$
Or, $x - y = 40$ (3)
Substituting $x - y = 40$ in (2), $40 : x = 8 : y$
By Art. 317, $5 : x = 1 : y$
Or, $x = 5y$ (5)
From (3) and (5) we find $x = 50$ and $y = 10$.

14. The square of the difference of two numbers is 9; and the difference of their cubes is to the cube of their difference as 43 : 3. What are the numbers?

15. The mean proportional between two numbers is 6; and the cube of the sum is to the sum of the cubes as 100 : 73. What are the numbers?

16. Find two numbers whose sum is 14, and whose product is to the sum of their squares as 10 : 29.

17. The difference of two numbers is 5; and the square of their difference is to the difference of their squares as 5 : 11. Find the two numbers.

18. As two boys were talking of their ages, they found that the product of the numbers representing their ages in years was 378; and the difference of the cubes of these same numbers was to the cube of their difference as 127 : 1. What was the age of each?

VARIATION.

325. A **Variable** is a number which may take a series of different values.

326. A **Constant** is a number whose value is fixed.

327. When the successive values of a variable constantly approach, by some fixed law, a constant, so that the difference between the variable and the constant may become as small as we please without actually reaching it, the constant is called the *limit* of the variable.

Thus, the continuous series,

$$0.6, \qquad 0.66, \qquad 0.666, \qquad 0.6666, \ldots\ldots,$$

is approaching its *limit*, the *constant* $\frac{2}{3}$.

328. A number is said to *vary* as another, if the two numbers are so related that, if one changes, the other changes in the same ratio. If they both increase in the same ratio, the variation is *direct;* but if one decreases in the same ratio as the other increases, the variation is *inverse.* For example, the number of dollars paid for a piece of work varies *directly* as the amount of work; but the number of days it takes to do a piece of work varies *inversely* as the number of men employed.

329. A number varies jointly as two or more others, if it changes as the product of these others change. For example, the number of dollars paid for a piece of work varies as the product of the number of men by the number of days they work.

330. Variation is denoted by the sign $\propto$ (read, varies as); thus, $a \propto b$ signifies that a varies directly as b; and $a \propto \frac{1}{b}$ signifies that a varies inversely as b.

NOTE. The word *directly* is usually omitted.

331. A variation is a proportion, and is the same as the statement in Art. 315. Thus, $a \propto b$ means that $a : b = ma : mb$; that is, any multiple of a is to the same multiple of b as a is to b; or, $a \propto b$ means that a, or any multiple of a, is a definite number of times b, or the same multiple of b; that is, if $a \propto b$, $a = nb$, and $ma = n(mb)$.

332. The statement that $a = nb$ if $a \propto b$, is not true of *quantities.* $\$a \propto b$ days, must mean $\$a : \$ma = b$ days : mb days, and it is not true that $\$a = nb$ days. It is true that the *number* of dollars varies as the *number* of days.

THEOREMS.

In the Theorems, m and n are constants.

333. *If $a \propto b$, and $b \propto c$, then $a \propto c$.*

For
$$a = mb, \text{ and } b = nc$$
$$\therefore a = mnc$$
$$\therefore a \propto c$$

334. *If $a \propto c$, and $b \propto c$, then $a \pm b \propto c$,* and $\sqrt{ab} \propto c$.

For
$$a = mc, \text{ and } b = nc$$
$$\therefore a \pm b = (m \pm n)c$$
$$\therefore a \pm b \propto c$$

Also,
$$\therefore \quad ab = mnc^2$$
$$\therefore \quad \sqrt{ab} = c\sqrt{mn}$$
$$\therefore \quad \sqrt{ab} \propto c$$

335. *If* $a \propto bc$, *then* $b \propto \frac{a}{c}$, *and* $c \propto \frac{a}{b}$.

For
$$a = mbc$$
$$\therefore \quad \frac{a}{c} = mb, \text{ and } \frac{a}{b} = mc$$
$$\therefore \quad b \propto \frac{a}{c}, \text{ and } c \propto \frac{a}{b}$$

336. *If* $a \propto b$, *and* $c \propto d$, *then* $ac \propto bd$.

For
$$a = mb, \text{ and } c = nd$$
$$\therefore \quad ac = mnbd$$
$$\therefore \quad ac \propto bd$$

337. *If* $a \propto b$, *then* $a^n \propto b^n$.

For
$$a = mb$$
$$\therefore \quad a^n = m^n b^n$$
$$\therefore \quad a^n \propto b^n$$

338. *If* $a \propto b$ *when* c *is constant, and* $a \propto c$ *when* b *is constant, then* $a \propto bc$ *when both* b *and* c *are variable.*

Suppose, while c is constant, a changes to a' and b to b';

then
$$\frac{a}{a'} = \frac{b}{b'} \qquad (1)$$

Then, suppose c changes to c', causing a' to change to a'';

then
$$\frac{a'}{a''} = \frac{c}{c'} \qquad (2)$$

Multiplying (1) and (2) together,
$$\frac{a}{a'} \times \frac{a'}{a''} = \frac{bc}{b'c'}$$
$$\therefore \quad \frac{a}{a''} = \frac{bc}{b'c'}$$
$$\therefore \quad a \propto bc$$

339. EXERCISES IN VARIATION.

1. If $x \propto y$, and, when $x = 5$, $y = 20$, what is the value of x when $y = 100$? Ans. $x = 25$.

2. If $a \propto \frac{1}{b}$, then $a\,b$ is constant.

3. If $a \propto \frac{1}{b}$, then $b \propto \frac{1}{a}$.

4. If $a \propto \frac{1}{b}$, and $b \propto \frac{1}{c}$, what is the relation of a to c?

5. If $a \propto b$, then $a\,x \propto b\,x$, and $\frac{a}{x} \propto \frac{b}{x}$.

6. If $x \propto \frac{1}{y}$, and, when $x = 25$, $y = 4$, what is x when $y = 20$?

7. If $a \propto b$ when c is constant, and $a \propto \frac{1}{c}$ when b is constant, prove that $a \propto \frac{b}{c}$.

8. If 4, 3, and 2 are simultaneous values of a, b, and c in Example 7, what is the value of a when $b = 3$ and $c = 1$?

9. If 8 men earn \$400 in 5 weeks, find by Example 7 how many weeks it will take 6 men to earn \$240?

10. The area of a triangle is measured by the product of its base and altitude. Prove that in triangles of equal area the bases vary inversely as the altitudes.

11. If $x^3 \propto y^2$, and $x = 3$ when $y = 4$, find the equation between x and y.

12. If $x \propto y$, and $x = \frac{4}{5}$ when $y = \frac{5}{4}$, what is x when $y = 25$?

13. If $a^2 - b^2 \propto c^2$, and, when $a = 5$ and $b = 3$, $c = 2$, find the equation between a, b, and c, and prove that b is a mean proportional between $a + 2\,c$ and $a - 2\,c$.

CHAPTER XXIII.

PROGRESSIONS.

340. A **Progression** is a series of numbers which increase or decrease according to some fixed law.

341. The **Terms** of a series are the successive numbers that form the series. The first and last terms are called the *extremes*, and the others the *means*.

ARITHMETIC PROGRESSION.

342. An **Arithmetic Progression** (A. P.) is a series in which each term, except the first, is obtained from the preceding by the addition of a constant number called the *common difference*. Thus, each of the following series is an arithmetic progression.

$$\begin{array}{ccccc}
3, & 6, & 9, & 12, & 15, \ldots \\
27, & 23, & 19, & 15, & 11, \ldots \\
-2, & -4, & -6, & -8, & -10, \ldots \\
a, & a+d, & a+2d, & a+3d, & a+4d, \ldots
\end{array}$$

The first is an *ascending progression*, the second a *descending progression*.

The common differences of the series are, respectively, 3, -4, -2, and d.

The common difference of an arithmetic progression can be found by subtracting any term from that which immediately follows it.

343. In Arithmetic Progression there are five elements, any three of which being given, the other two can be found.

1. The first term.
2. The last term.
3. The common difference.
4. The number of terms.
5. The sum of all the terms.

344. In Arithmetic Progression there are twenty possible cases. In discussing this subject we shall let

$a =$ the first term,
$l =$ the last term,
$d =$ the common difference,
$n =$ the number of terms,
$s =$ the sum of all the terms.

Case I.

345. The First Term, Common Difference, and Number of Terms given, to find the Last Term.

In this Case, a, d, and n are given, and l is required.

The successive terms of the series are

$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad a + 4d, \ldots$$

that is, the coefficient of d in each term is one less than the number of that term, counting from the left; therefore, the last or nth term in the series is

$$a + (n - 1)d$$

or

$$l = a + (n - 1)d$$

in which the series is ascending or descending according as d is positive or negative. Hence,

Rule.

To the first term add the product formed by multiplying the common difference by the number of terms less one.

1. Given $a = 2$, $d = 3$, and $n = 17$, to find l.

$$l = a + (n - 1)\,d = 2 + (17 - 1)\,3 = 50 \quad \text{Ans.}$$

2. Given $a = -7$, $d = 3$, and $n = 8$, to find l.

3. Given $a = -\frac{3}{4}$, $d = -\frac{7}{8}$, and $n = 25$, to find l.

4. Given $a = \frac{1}{2}$, $d = \frac{1}{8}$, and $n = 20$, to find l.

Case II.

346. The Extremes and the Number of Terms given, to find the Sum of the Series.

In this Case, a, l, and n are given, and s is required.

Now $\quad s = a + (a + d) + (a + 2d) + (a + 3d) + \cdots + l$

or, reversing the series, $\quad s = l + (l - d) + (l - 2d) + (l - 3d) + \cdots + a$

Adding these together, $\quad 2s = (a + l) + (a + l) + (a + l) + (a + l) + \cdots + (a + l)$

And since $(a + l)$ is to be taken as many times as there are terms,

hence $\quad 2s = n\,(a + l)$

or, $\quad s = \frac{n}{2}\,(a + l)$. Hence,

Rule.

Find one half the product of the sum of the extremes and the number of terms.

Note. If in place of the last term the common difference is given, the last term must first be found by the Rule in Case I.

1. Given $a = 2$, $l = 50$, and $n = 17$, to find s.

$$s = \frac{n}{2}\,(a + l) = \frac{17}{2}\,(2 + 50) = 442 \quad \text{Ans.}$$

2. Given $a = 5$, $l = 19$, and $n = 7$, to find s.
Ans. $s = 84$.

3. Given $a = -10$, $d = -2$, and $n = 6$, to find s.

4. Given $a = 5\frac{1}{2}$, $d = 1\frac{1}{4}$, and $n = 17$, to find s

5. Given $a = \frac{1}{4}$, $d = -\frac{1}{2}$, and $n = 21$, to find s.

Case III.

347. The Extremes and Number of Terms given, to find the Common Difference.

In this Case, a, l, and n are given, and d is required.

From Case I. we have $\quad l = a + (n-1)\,d$

Transposing and reducing, $\quad d = \dfrac{l-a}{n-1}.\quad$ Hence,

Rule.

Divide the last term minus the first term by the number of terms less one, and the quotient will be the common difference.

1. Given $a = 3$, $l = 23$, and $n = 6$, to find d.

$$d = \frac{l-a}{n-1} = \frac{23-3}{6-1} = 4 \quad \text{Ans.}$$

2. Given $a = 7$, $l = 37$, and $n = 11$, to find d.
3. Given $a = 9$, $l = -6$, and $n = 16$, to find d.
4. Given $a = \frac{13}{27}$, $l = \frac{1}{3}$, and $n = 5$, to find d.

Note 1. This rule enables us to insert any number of arithmetic means between two given numbers; for the number of terms is two greater than the number of means. Hence, if $m =$ the number of means, $m + 2 = n$, or $m + 1 = n - 1$, and $d = \dfrac{l-a}{m+1}\cdot$ Having found the common difference, the means are found by adding the common difference once, twice, &c., to the first term.

5. Find 6 arithmetic means between 4 and 39.
6. Find 4 arithmetic means between 17 and 52.
7. Find 6 arithmetic means between 2 and -26.

Note 2. When $m = 1$, the formula becomes $d = \dfrac{l-a}{2}.$ Adding a to each member,

$$a + d = \frac{l-a}{2} + a = \frac{l+a}{2}$$

But $a + d$ is the second term of a series whose first term is a and common difference d, or the arithmetic mean of the series a, $a + d$, $a + 2\,d$. Hence, *the arithmetic mean between two numbers is one half of their sum.*

8. Find the arithmetic mean between 8 and 10. Ans. 9.

9. Find the arithmetic mean between $\frac{1}{7}$ and $\frac{5}{7}$.

10. Find the arithmetic mean between 7 and -7.

348. From the formulas established in Arts. 345 and 346, viz.:

$$l = a + (n-1)\,d \qquad (1)$$

$$s = \frac{n}{2}\,(a + l) \qquad (2)$$

can be derived formulas for all the Cases in Arithmetic Progression.

From (1) we can obtain the value of any one of the four numbers, l, a, n, or d, when the other three are given; and from (2), the value of s, n, a, or l, when the other three are given. Formulas for the remaining twelve Cases are obtained by combining the two formulas (1) and (2), so as to eliminate that one of the two unknown numbers whose value is not sought.

1. Find the formula for the value of n, when a, d, and s are given.

$$l = a + (n-1)\,d \quad (1) \qquad\qquad s = \frac{n}{2}\,(a + l) \quad (2)$$

$$l\,n = a\,n + d\,n^2 - d\,n \quad (3) \qquad\qquad 2\,s - a\,n = l\,n \quad (4)$$

$$a\,n + d\,n^2 - d\,n = 2\,s - a\,n \qquad (5)$$

$$n^2 - \left(\frac{d - 2\,a}{d}\right) n = \frac{2\,s}{d} \qquad (6)$$

$$n = \frac{d - 2\,a \pm \sqrt{(d - 2\,a)^2 + 8\,d\,s}}{2\,d} \qquad (7)$$

To obtain the formula required in this example, l must be eliminated from (1) and (2). From (1) and (2) we obtain (3) and (4). Placing these two values of $l\,n$ equal to each other, we form (5), which reduced gives (7), or the value of n in known numbers.

349. The twenty Cases appear in the following table.

From Nos. 1 and 9 let the pupil obtain the other formulas.

No.	Given.	Required.	Results.
1	$a\,d\,n$	l	$l = a + (n-1)\,d.$
2	$a\,d\,s$		$l = -\frac{1}{2}d \pm \sqrt{2\,d\,s + (a - \frac{1}{2}d)^2}.$
3	$a\,n\,s$		$l = \frac{2\,s}{n} - a.$
4	$d\,n\,s$		$l = \frac{s}{n} + \frac{(n-1)\,d}{2}.$
5	$d\,n\,l$	a	$a = l - (n-1)\,d.$
6	$d\,l\,s$		$a = \frac{1}{2}d \pm \sqrt{(\frac{1}{2}d + l)^2 - 2\,d\,s}.$
7	$n\,l\,s$		$a = \frac{2\,s}{n} - l.$
8	$d\,n\,s$		$a = \frac{s}{n} - \frac{(n-1)\,d}{2}.$
9	$a\,d\,n$	s	$s = \frac{1}{2}n\,[2\,a + (n-1)\,d].$
10	$a\,d\,l$		$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2\,d}.$
11	$a\,n\,l$		$s = \frac{n}{2}(a + l).$
12	$d\,n\,l$		$s = \frac{1}{2}n\,[2\,l - (n-1)\,d].$
13	$a\,n\,l$	d	$d = \frac{l-a}{n-1}.$
14	$a\,n\,s$		$d = \frac{2\,(s - a\,n)}{n\,(n-1)}.$
15	$a\,l\,s$		$d = \frac{l^2 - a^2}{2\,s - l - a}.$
16	$n\,l\,s$		$d = \frac{2\,(n\,l - s)}{n\,(n-1)}.$
17	$a\,d\,l$	n	$n = \frac{l-a}{d} + 1.$
18	$a\,d\,s$		$n = \frac{d - 2\,a \pm \sqrt{(d - 2\,a)^2 + 8\,d\,s}}{2\,d}.$
19	$a\,l\,s$		$n = \frac{2\,s}{l+a}.$
20	$d\,l\,s$		$n = \frac{2\,l + d \pm \sqrt{(2\,l + d)^2 - 8\,d\,s}}{2\,d}.$

350. To find any one of the five elements when three others are given, we *substitute the given values in that formula whose first member is the required term, and whose second contains the three given terms.*

1. Given $d = 2$, $l = 21$, and $s = 120$, to find a.

$$a = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2} + l\right)^2 - 2ds} \qquad (1)$$

$$a = \frac{2}{2} \pm \sqrt{\left(\frac{2}{2} + 21\right)^2 - 2 \cdot 2 \cdot 120} \qquad (2)$$

$$a = 3, \text{ or } -1 \qquad (3)$$

In the table, Art. 349, we find (1), the required formula; substituting the given values of d, l, and s, we obtain (2), which reduced gives (3), or $a = 3$, or -1.

NOTE 1. If $a = 3$, $n = 10$; but if $a = -1$, $n = 12$.

2. Given $d = 4$, $l = 36$, and $s = 120$, to find n.
Ans. $n = 4$, or 15.

NOTE 2. When $n = 4$, $a = 24$; but when $n = 15$, $a = -20$.

3. Given $d = \frac{1}{12}$, $n = 8$, and $s = 3$, to find l.
Ans. $l = \frac{2}{3}$.

4. Given $d = 3$, $n = 9$, and $a = -12$, to find s.

5. Given $a = 5\frac{1}{2}$, $l = 25\frac{1}{2}$, and $s = 263\frac{1}{2}$, to find d.

351. MISCELLANEOUS EXAMPLES.

Find the 100th term of the following series:

1. 3, 5, 7, &c. 2. $\frac{1}{9}$, $\frac{1}{3}$, $\frac{5}{9}$, &c. 3. $a + b$, a, $a - b$, &c.

Find the sum of the following series:

4. -3, 1, 5, . . . to 17 terms.

5. a, $3a$, $5a$, . . . to a terms. Ans. a^3.

6. $\frac{1}{\sqrt{2}+1}$, $\sqrt{2}$, $\frac{1}{\sqrt{2}-1}$, . . . to 7 terms.

Ans. $7(\sqrt{2}+2)$.

7. $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, . . . to n terms. Ans. $\frac{n-1}{2}$.

8. The 3d term of an A. P. is 10, and the 14th term is 54. Find the 20th term. Ans. 78.

9. Which term of the series $\frac{7}{6}$, $\frac{4}{3}$, $\frac{3}{2}$, etc. is 18?

Ans. 102d.

10. Insert 10 arithmetic means between $5a - 6b$, and $5b - 6a$. Ans. $4a - 5b$, $3a - 4b$, etc.

11. Show that the sum of any number of consecutive odd numbers, beginning with unity, is a square number.

12. Find the sum of 15 terms of an A. P. of which the 8th is 6.

13. Find the sum of all the numbers between 100 and 500 which are divisible by 3.

14. How many terms of the series 24, 21, 18, etc. must be taken in order that the sum may be 105? Ans. 7 or 10.

15. How many strokes do the clocks of Italy, which strike up to 24, make in one revolution of the index?

16. One hundred stones are placed on the ground at a distance of 2 feet from each other, the first 20 yards from the basket. How far does a person travel who starts from the basket and brings them one by one to it? Ans. $6\frac{1}{44}$ miles.

17. If the nth term is $\frac{3n-1}{6}$, and $s = \frac{n}{12}(3n+1)$, find the series. Ans. $\frac{1}{3}$, $\frac{5}{6}$, $\frac{8}{6}$, etc.

18. An A. P. consists of 21 terms; the sum of the three terms in the middle is 129, and of the last three is 237. Find the series. Ans. 3, 7, 11, . . . 83.

19. If the sum of the second and eleventh terms is equal to three times that of the fourth and fifth, and the eighth term is 12, find the series. Ans. $-\frac{20}{3}, -\frac{12}{3}, -\frac{4}{3}, \frac{4}{3}$, etc.

20. The sum of three numbers in A. P. is 9, and the sum of their squares is 29. What are the numbers?

(Let $x - y$, x, $x + y$, represent the numbers.)

21. The sum of three numbers in A. P. is 33, and their product is 792. What are the numbers.

22. There are four numbers in A. P.; the sum of the squares of the extremes is 272, and the sum of the squares of the means is 208. What are the numbers?

(Let $x - 3y$, $x - y$, $x + y$, $x + 3y$, represent the numbers.)

23. The base of a right triangle is 12, and its sides are in A. P. Find the other sides.

24. The product of five numbers in A. P. is 10395, and their sum is 35. What are the numbers?

Ans. 3, 5, 7, 9, 11.

25. The three digits of a certain number are in A. P.; if the number is divided by the sum of the digits in the units' and tens' place, the quotient is 107. If 396 is subtracted from the number, the order of its digits will be reversed. Required the number.

26. A and B have to walk a distance of 27 miles; A starts at $2\frac{1}{2}$ miles an hour, and increases his pace by a quarter of a mile every hour; B starts at 5 miles an hour, but falls off at the rate of half a mile every hour. Find which will finish the distance first, and by what length of time.

Ans. A, by an hour.

27. A body falls through a space of 4.9 meters the first second, and in each succeeding second 9.8 meters more than in the next preceding one. How far will it fall in 20 seconds?

28. A man saves every year $25, which he puts at interest at the rate of 4 per cent a year. How long will it take for the interest to amount to $91?

29. Divide unity into four parts in A. P., of which the sum of the cubes shall be $\frac{1}{10}$.

30. A and B jointly owe $23661, of which A offers to pay every day $20, if B will pay $1 the first day, $2 the second, and so on in A. P. How many days will it take to pay the debt, and how much will each pay?

Ans. 198 days; A $3960; B $19701.

GEOMETRIC PROGRESSION.

352. A **Geometric Progression** (G. P.) is a series in which each term, except the first, is obtained by multiplying the preceding term by a constant number, called the *ratio.*

Thus, each of the following series forms a Geometric Progression.

$$\begin{array}{rrrr} 3, & 6, & 12, & 24, \ldots \\ 72, & 36, & 18, & 9, \ldots \\ 1, & -\frac{1}{3}, & \frac{1}{9}, & -\frac{1}{27}, \ldots \\ a, & ar, & ar^2, & ar^3, \ldots \end{array}$$

The first is an *ascending* progression, the second a *descending* progression.

The ratios of the series are respectively 2, $\frac{1}{2}$, $-\frac{1}{3}$, and r.

The *ratio* of a geometric progression can be found by dividing *any* term by that which immediately precedes it.

353. In Geometric Progression there are five elements, any three of which being given, the other two can be found. These elements are the same as in Arithmetic Progression, except that in place of the common difference we have *the ratio.*

354. In Geometric Progression there are twenty possible cases. In discussing these cases we shall preserve the same notation as in Arithmetic Progression, except that, instead of $d =$ the common difference, we shall use $r =$ the ratio.

CASE I.

355. The First Term, Ratio, and Number of Terms given, to find the Last Term.

In this Case, a, r, and n are given, and l required.

The successive terms of the series are

$$a,\ a\,r,\ a\,r^2,\ a\,r^3,\ a\,r^4, \text{ etc.}$$

That is, each term is the product of the first term and that power of the ratio which is one less than the number of the term; therefore the last or nth term in the series is

$$a\,r^{n-1}, \quad \text{or} \quad l = a\,r^{n-1}. \quad \text{Hence,}$$

Rule.

Multiply the first term by that power of the ratio whose index is one less than the number of terms.

1. Given $a = 7$, $r = 2$, and $n = 8$, to find l.
$$l = a\,r^{n-1} = 7 \times 2^7 = 896 \quad \text{Ans.}$$

2. Given $a = 3$, $r = 2$, and $n = 10$, to find l.
Ans. $l = 1536$.

3. Given $a = 3$, $r = \frac{1}{3}$, and $n = 6$, to find l.

4. Given $a = -5$, $r = -3$, and $n = 7$, to find l.

5. Given $a = -\frac{1}{3}$, $r = \frac{3}{2}$, and $n = 8$, to find l.

Case II.

356. The Extremes and the Ratio given, to find the Sum of the Series.

In this Case, a, l, and r are given, and s is required.

Now $$s = a + ar + ar^2 + ar^3 + \cdots + l \qquad (1)$$

Multiplying (1) by r, $$rs = ar + ar^2 + ar^3 + \cdots + l + lr \qquad (2)$$

Subtracting (1) from (2), $$rs - s = lr - a$$

Whence, $$s = \frac{lr - a}{r - 1}.$$ Hence,

Rule.

Multiply the last term by the ratio, from the product subtract the first term, and divide the remainder by the ratio less one.

1. Given $a = 5$, $l = 320$, and $r = 2$, to find s.

$$s = \frac{lr - a}{r - 1} = \frac{320 \times 2 - 5}{2 - 1} = 635 \quad \text{Ans.}$$

2. Given $a = 4$, $l = 78732$, and $r = 3$, to find s.

Ans. $s = 118096$.

3. Given $a = 5$, $l = -640$, and $r = -2$, to find s.

Case III.

357. The First Term, Ratio, and Number of Terms given, to find the Sum of the Series.

In this Case, a, r, and n are given, and s required.

The last term can be found by Case I., and then the sum of the series by Case II. Or better, since

$$l = ar^{n-1}$$
$$lr = ar^n$$

Substituting this value of lr in the formula in Case II., we have

$$s = \frac{r^n - 1}{r - 1} \times a.$$ Hence,

Rule.

From the ratio raised to a power whose index is equal to the number of terms subtract one, divide the remainder by the ratio less one, and multiply the quotient by the first term.

1. Given $a = 7$, $r = 2$, and $n = 6$, to find s.

$$s = \frac{r^n - 1}{r - 1} \times a = \frac{2^6 - 1}{2 - 1} \times 7 = 441 \quad \text{Ans.}$$

2. Given $a = 5$, $r = 4$, and $n = 9$, to find s.

Ans. $s = 436905$.

3. Given $a = 2$, $r = -\frac{1}{2}$, and $n = 8$, to find s.

4. Given $a = \frac{2}{3}$, $r = -\frac{3}{2}$, and $n = 7$, to find s.

5. Given $a = 2$, $r = -\frac{3}{2}$, and $n = 6$, to find s.

358. In a geometric series whose ratio is a proper fraction, the greater the number of terms, the less numerically the last term. If the number of terms is infinite, the last term must be infinitesimal; and in finding the sum of such a series the last term may be considered as nothing. Therefore, when the number of terms is infinite, the formula

$$s = \frac{lr - a}{r - 1} \text{ becomes}$$

$$s = \frac{-a}{r - 1} = \frac{a}{1 - r}.$$

Hence, to find the sum of a geometric series whose ratio is a proper fraction and number of terms infinite,

Rule.

Divide the first term by one minus the ratio.

1. Find the sum of the series 12, 6, 3, etc. to infinity.

$$s = \frac{a}{1 - r} = \frac{12}{1 - \frac{1}{2}} = 24 \quad \text{Ans.}$$

2. Find the sum of the series 5, 1, $\frac{1}{5}$, etc. to infinity.

Ans. $\frac{25}{4}$.

3. Find the sum of the series a, b, $\frac{b^2}{a}$, etc., if $b < a$, to infinity. Ans. $\frac{a^2}{a-b}$.

4. Find the sum of the series 0.9, 0.03, 0.001, etc. to infinity.

5. Find the value of the decimal 0.3333, etc. to infinity.

(This decimal can be written $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$, etc.)

Case IV.

359. The Extremes and Number of Terms given, to find the Ratio.

In this Case, a, l, and n are given, and r is required.

From Case I. $l = ar^{n-1}$

Whence, $r = \sqrt[n-1]{\frac{l}{a}}$. Hence,

Rule.

Divide the last term by the first, and extract that root of the quotient whose index is one less than the number of terms.

1. Given $a = 2$, $l = 162$, and $n = 5$, to find r.

$$r = \sqrt[n-1]{\frac{l}{a}} = \sqrt[4]{\frac{162}{2}} = 3 \quad \text{Ans.}$$

2. Given $a = 2$, $l = 10\frac{1}{8}$, and $n = 5$, to find r. Ans. $r = \frac{3}{2}$.

3. Given $a = -\frac{1}{5}$, $l = \frac{1}{3125}$, and $n = 6$, to find r.

Note 1. This rule enables us to insert any number of geometric means between two numbers; for the number of terms is two greater than the number of means. Hence, if $m =$ the number of means, $m + 2 = n$, or $m + 1 = n - 1$; and $r = \sqrt[m+1]{\frac{l}{a}}$. Having found the ratio, the means are found by multiplying the first term by the *ratio*, by its *square*, its *cube*, &c.

4. Find three geometric means between 5 and 405.

$$r = \sqrt[m+1]{\frac{l}{a}} = \sqrt[4]{\frac{405}{5}} = \sqrt[4]{81} = 3$$

Ans. 15, 45, 135.

5. Find four geometric means between 160 and 5.

6. Find three geometric means between 7 and $35\frac{7}{16}$.

NOTE 2. When $m = 1$, the formula becomes

$$r = \sqrt{\frac{l}{a}}$$

Multiplying by a, $$a\,r = a\sqrt{\frac{l}{a}} = \sqrt{a\,l}.$$

But $a\,r$ is the second term of a series whose first term is a and ratio r; or the geometric mean of the series a, $a\,r$, $a\,r^2$. Hence, *the geometric mean between two numbers is the square root of their product.*

7. Find the geometric mean between 3 and 27. Ans. 9.

8. Find the geometric mean between $\frac{1}{3}$ and 2187.

9. Find the geometric mean between $-\frac{1}{5}$ and $-\frac{1}{1125}$.

360. From the formulas established in Arts. 355 and 356,

$$l = a\,r^{n-1} \tag{1}$$

$$s = \frac{l\,r - a}{r - 1} \tag{2}$$

can be derived formulas for all the Cases in Geometric Progression.

From (1) we can obtain the value of any one of the four numbers, l, a, n, or r, when the other three are given; from (2), the value of s, l, r, or a, when the other three are given. Formulas for the remaining twelve Cases are obtained by combining the formulas (1) and (2) so as to eliminate that one of the two unknown numbers whose value is not sought.

1. Find the formula for the value of s, when l, n, and r are given.

From (1), $$\frac{l}{r^{n-1}} = a$$

Substituting this value of a in (2), $$s = \frac{l\,r - \dfrac{l}{r^{n-1}}}{r - 1}$$

or, $$s = \frac{(r^n - 1)\,l}{(r - 1)\,r^{n-1}}$$

361. The twenty Cases appear in the following table.

From Nos. 1 and 9 let the pupil obtain the other formulas, except those named in the note on the next page.

No.	Given.	Required.	Results.
1	$a\,r\,n$	l	$l = a r^{n-1}$.
2	$a\,r\,s$		$l = \dfrac{a + (r-1)s}{r}$.
3	$r\,n\,s$		$l = \dfrac{(r-1)s r^{n-1}}{r^n - 1}$.
4	$a\,n\,s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.
5	$r\,n\,l$	a	$a = \dfrac{l}{r^{n-1}}$.
6	$r\,n\,s$		$a = \dfrac{(r-1)s}{r^n - 1}$.
7	$r\,l\,s$		$a = rl - (r-1)s$.
8	$n\,l\,s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$.
9	$a\,r\,n$	s	$s = \dfrac{a(r^n - 1)}{r-1}$.
10	$a\,r\,l$		$s = \dfrac{lr - a}{r-1}$.
11	$a\,n\,l$		$s = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$.
12	$r\,n\,l$		$s = \dfrac{(r^n - 1)l}{(r-1)r^{n-1}}$.
13	$a\,n\,l$	r	$r = \sqrt[n-1]{\dfrac{l}{a}}$.
14	$a\,n\,s$		$r^n - \dfrac{s}{a}r + \dfrac{s-a}{a} = 0$.
15	$a\,l\,s$		$r = \dfrac{s-a}{s-l}$.
16	$n\,l\,s$		$r^n - \dfrac{s}{s-l}r^{n-1} + \dfrac{l}{s-l} = 0$.
17	$a\,r\,l$	n	$n = \dfrac{\log l - \log a}{\log r} + 1$.
18	$a\,r\,s$		$n = \dfrac{\log[a + (r-1)s] - \log a}{\log r}$.
19	$a\,l\,s$		$n = \dfrac{\log l - \log a}{\log(s-a) - \log(s-l)} + 1$.
20	$r\,l\,s$		$n = \dfrac{\log l - \log[lr - (r-1)s]}{\log r} + 1$.

NOTE. The four formulas for the value of n cannot be obtained or used without a knowledge of logarithms; and four others, viz. 4, 8, 14, and 16, when n exceeds 2, cannot be reduced without a knowledge of equations that cannot be reduced by any rules given in this book.

362. To find any one of the five elements when three others are given, we *substitute the given values in that formula whose first member is the required term, and whose second contains the three given terms.*

1. Given $r = 3$, $n = 5$, and $s = 726$, to find l.

$$l = \frac{(r-1)\,s\,r^{n-1}}{r^n - 1} \qquad (1)$$

$$l = \frac{(3-1)\,(726)\,(3)^{5-1}}{3^5 - 1} = 486 \qquad (2)$$

In the table, Art. 361, we find (1), the required formula, and, substituting the given values of r, n, and s, we obtain (2), or $l = 486$.

2. Given $a = 1$, $n = 8$, and $l = 128$, to find s.

Ans. $s = 255$.

3. Find the 7th and 11th terms of the series $64, -32, \ldots$

4. Find the 4th and 8th terms of the series $0.008, 0.04, \ldots$

363. MISCELLANEOUS EXAMPLES.

Find the last term in the following series:

1. $x, 1, \frac{1}{x}, \ldots$ to 30 terms. Ans. $\frac{1}{x^{28}}$.

2. x, x^3, x^5, to p terms. Ans. x^{2p-1}.

Find the sum of:

3. $16.2, \ 5.4, \ 1.8, \ldots$ to 7 terms.

4. $1, 5, 25, \ldots$ to p terms. Ans. $\frac{1}{4}(5^p - 1)$.

5. $a, \frac{a}{r}, \frac{a}{r^2}, \ldots$ to n terms.

6. $\frac{8}{5}, -1, \frac{5}{8}, \ldots$ to infinity.

7. $3^{-1}, 3^{-2}, 3^{-3}, \ldots$ to infinity.

8. $\sqrt{\frac{3}{2}}, \frac{1}{3}\sqrt{2}, \frac{2}{9}\sqrt{\frac{2}{3}}, \ldots$ to infinity.

Ans. $\dfrac{9(3\sqrt{6} + 2\sqrt{2})}{46}$.

9. Insert 3 geometric means between 486 and 6.

10. Insert 5 geometric means between $\frac{32}{81}$ and $4\frac{1}{2}$.

11. The 5th term of a G. P. is $\frac{8}{9}$, and the 7th term $\frac{32}{81}$. Determine the series. Ans. $\frac{9}{2}, \pm 3, 2, \pm \frac{4}{3}, \ldots$

12. Find a G. P. whose first term is unity, and whose third term is $\frac{1}{25}$.

13. The sum of an infinite series in G. P. is 12, and the second term is 27 times the fifth term. Find the series.

14. The sum of an infinite series in G. P. is $\frac{2}{3}$, and the sum of the first two terms is $\frac{14}{25}$. What is the series?

15. What is the amount at compound interest of $500 for 5 years at 6 %?

16. Find the amount at compound interest of $\$p$ for n years at r per cent.

17. From a vessel containing $182\frac{1}{4}$ centiliters of alcohol a chemist draws daily a fixed quantity, and replaces it with water. At the end of 6 days there are only 16 centiliters of alcohol in the vessel. How many centiliters does he draw a day. Ans. $60\frac{3}{4}$.

18. A certain number is formed of three digits that are in G. P. Now twice the digit in the hundreds' place is equal to the difference between the digits in the tens' and the units' place; and if 594 is added to the number, the order of the digits will be reversed. What is the number?

Let x, xy, xy^2, represent the digits.

NOTE. It will often be best to represent a G. P. as follows:

$$x^2, \quad xy, \quad y^2, \quad \text{for three terms;}$$

$$\frac{x^2}{y}, \quad x, \quad y, \quad \frac{y^2}{x}, \text{ for four terms;}$$

$$\frac{x^3}{y}, \quad x^2, \quad xy, \quad y^2, \quad \frac{y^3}{x}, \text{ for five terms.}$$

19. There are three numbers in geometric progression whose sum is 63; and the sum of the extremes is to the mean as 17 : 4. What are the numbers?

20. There are five numbers in geometric progression; the sum of the first four is 40, and the sum of the last four 120. What are the numbers?

21. The sum of the squares of three numbers in geometric progression is 819; and the sum of the extremes is 21 more than the mean. What are the numbers?

22. There are four numbers in geometric progression whose continued product is 729; and the sum of the series is to the sum of the means as 10 : 3. What are the numbers?

23. Of four numbers in geometric progression the sum of the first and third is 51; and the difference of the means is to the difference of the extremes as 4 : 21. What are the numbers?

HARMONIC PROGRESSION.

364. An **Harmonic Progression** (H. P.) is a series of numbers whose reciprocals are in arithmetic progression.

Thus, $a, \; b, \; c, \ldots$ are in H. P.

if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \ldots$ are in A. P.

365. There is no general formula for the *sum* of an H. P., but many problems with respect to such a series may be solved by inverting the terms and proceeding as in an A. P., and then inverting back again.

THEOREMS.

366. *If a, b, c, are in H. P., then $a : c = a - b : b - c$.*

Since $\quad a, \quad b, \quad c,$ are in H. P.,

$$\frac{1}{a}, \quad \frac{1}{b}, \quad \frac{1}{c}, \text{ are in A. P.}$$

$$\therefore \quad \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

or,
$$\frac{a-b}{a} = \frac{b-c}{c}$$

or,
$$a : c = a - b : b - c$$

367. *The harmonic mean between two numbers is twice their product divided by their sum.*

Let a, b, be the two numbers, and H their harmonic mean.

Then
$$\frac{1}{a}, \frac{1}{H}, \frac{1}{b}, \text{ are in A. P.}$$

$$\therefore \quad \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}$$

$$\frac{2}{H} = \frac{1}{a} + \frac{1}{b}$$

$$H = \frac{2ab}{a+b}$$

368. *The geometric mean between two numbers is the geometric mean between the arithmetic and harmonic means of the same numbers.*

Let A, G, H, represent, respectively, the arithmetic, geometric, and harmonic means between a and b; then

Art. 347, Note 2, $\qquad A = \dfrac{a+b}{2}$

Art. 359, Note 2, $\qquad G = \sqrt{ab}$

Art. 367, $\qquad H = \dfrac{2ab}{a+b}$

$$\therefore \quad A \times H = \frac{a+b}{2} \times \frac{2ab}{a+b} = ab$$

and
$$G^2 = ab$$

369. EXAMPLES.

1. The second term of an H. P. is 2, and the fourth term is 6. Find the series.

The 2d and 4th terms of the corresponding A. P. are $\frac{1}{2}$ and $\frac{1}{6}$, respectively. Therefore $a + d = \frac{1}{2}$, and $a + 3d = \frac{1}{6}$; whence $a = \frac{2}{3}$, and $d = -\frac{1}{6}$. Therefore the A. P. is $\frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \ldots$, and the H. P. is $\frac{3}{2}$, 2, 3, 6, ...

2. If a, b, c, are in H. P., prove that $\frac{a}{b+c}, \frac{b}{a+c}, \frac{c}{a+b}$, are also in H. P.

Since $\frac{1}{a}, \quad \frac{1}{b}, \quad \frac{1}{c}$, are in A. P.

$\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$, are in A. P.

$\therefore \quad 1 + \frac{b+c}{a}, \; 1 + \frac{a+c}{b}, \; 1 + \frac{a+b}{c}$, are in A. P.

$\therefore \quad \frac{b+c}{a}, \quad \frac{a+c}{b}, \quad \frac{a+b}{c}$, are in A. P.

Hence, $\frac{a}{b+c}, \quad \frac{b}{a+c}, \quad \frac{c}{a+b}$, are in H. P.

Find the arithmetic, harmonic, and geometric means between:

3. $\frac{1}{4}$ and $\frac{1}{10}$.

4. $x + y$ and $x - y$.

5. $\frac{1}{x+y}$ and $\frac{1}{x-y}$.

6. Insert three harmonic means between $2\frac{2}{5}$ and 12.
Ans. 3, 4, 6.

7. Continue to five terms the H. P. $1\frac{2}{3}$, 1, $\frac{5}{7}$.

8. The first term of an H. P. is unity, the third term is $\frac{1}{3}$. Find the tenth term. Ans. $\frac{1}{10}$.

9. If $b + c$, $a + c$, $a + b$, are in H. P., prove that a^2, b^2, c^2, are in A. P.

10. The difference of the arithmetic and harmonic means between two numbers is $1\frac{4}{5}$. Find the numbers, one being four times the other. Ans. 2, 8.

CHAPTER XXIV.

THE BINOMIAL THEOREM.

370. THE laws for the expansion of a binomial, as illustrated by actual multiplication in Art. 208, can be proved to be true for any index. The following is the proof when the index is any positive integer. Following the laws of Art. 208, we have

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{1\cdot 2}a^{m-2}b^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}a^{m-3}b^3 +, \text{ etc.}$$

Now, multiplying this equation by $a+b$, for the first member we have $(a+b)^{m+1}$; and for the second,

$$\begin{array}{l}
a^m + m\,a^{m-1}b + \dfrac{m(m-1)}{1\cdot 2}a^{m-2}b^2 + \dfrac{m(m-1)(m-2)}{1\cdot 2\cdot 3}a^{m-3}b^3 +, \text{ etc.}\\
a+b\\
\hline
a^{m+1} + m\,a^m b + \dfrac{m(m-1)}{1\cdot 2}a^{m-1}b^2 + \dfrac{m(m-1)(m-2)}{1\cdot 2\cdot 3}a^{m-2}b^3 +, \text{ etc.}\\
\qquad a^m b + \qquad m\,a^{m-1}b^2 + \qquad \dfrac{m(m-1)}{1\cdot 2}a^{m-2}b^3 +, \text{ etc.}\\
\hline
a^{m+1} + (m+1)\,a^m b + \dfrac{(m+1)m}{1\cdot 2}a^{m-1}b^2 + \dfrac{(m+1)m(m-1)}{1\cdot 2\cdot 3}a^{m-2}b^3 +, \text{ etc.}
\end{array}$$

Now let $m+1=n$, or $m=n-1$, and this product becomes

$$a^n + n\,a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 +, \text{ etc.}$$

That is, $(a+b)^m \times (a+b) = (a+b)^{m+1} = (a+b)^n$

$$= a^n + n\,a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 +, \text{ etc.}$$

and this is exactly the form that is obtained in applying the laws of Art. 208 directly to expanding $a+b$ to the nth power. Therefore, if the laws are true for the power whose index is m, they are true for the power whose index is $(m+1)$. These laws have been proved to

be true for the 5th power (§ 208); therefore they are true for the 6th; and therefore for the 7th; and so on.

NOTE. This method of proof is called *mathematical induction.*

371. If $-b$ is substituted for b, in the binomial $a+b$, the signs of the terms containing the odd powers of b will be $-$ (§ 208); that is, the second, fourth, sixth, etc. terms.

372. *To find any term in the expansion of a binomial.*

The indices in any term in the expansion of $(a+b)^n$ can be written at once from the last formula in Art. 370.

Thus, the indices in the rth form are, for a, $n-(r-1)=n+1-r$, and for b, $r-1$; and if the sign of b is $+$, the sign of the rth term is $+$, but if b is $-$, and $r-1$ is odd, that is, if r is even, the sign is $-$, otherwise $+$.

The coefficient of the 2d term is n

" " " 3d " $\dfrac{n(n-1)}{1\cdot 2}$

" " " 4th " $\dfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$

" " " 5th " $\dfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$

" " " 6th " $\dfrac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$

and so on. Hence,

The rth term of the nth power of $a+b$ is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1\cdot 2\cdot 3\ldots\ldots(r-1)}a^{n-r+1}b^{r-1}$$

1. Find the 6th term in $(a-b)^{18}$.

In this example $n=18$, and $r=6$. Hence, the index of a in the 6th term is $18-(6-1)=13$; and the index of b is 5. The sign of the 6th term is $-$; $n-r+2=14$; $r-1=5$; and the coefficient of the 6th term is

$$\frac{18\cdot 17\cdot \overset{2}{\cancel{16}}\cdot \cancel{15}\cdot 14}{1\cdot \cancel{2}\cdot \cancel{3}\cdot \cancel{4}\cdot \cancel{5}}=8568$$

Hence, the 6th term in $(a-b)^{18}=-8568\,a^{13}b^5$.

NOTE 1. In writing the formula for the coefficient, write for the denominator $1 \cdot 2 \cdot 3 \cdot 4 \ldots$ etc., with the last number $= r - 1$; then, for the numerator, over *each* number in the denominator write a series beginning with a number $= n$, and decreasing by 1.

Find the

2. 8th term in $(x - y)^{16}$.
5. 96th term of $(m - n)^{100}$.

NOTE 2. Notice that the 96th term of the 100th power is the 6th from the last term.

3. 7th term of $(a + b)^{15}$.
6. Middle term of $(x + y)^{14}$.
4. 8th term of $(x - y)^{20}$.
7. Middle terms of $(m - n)^{17}$.
8. 7th term in $(2x + 3y)^{11}$.

$$\frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(2x)^5(3y)^6 = 11 \cdot 7 \cdot 2^6 \cdot 3^7\, x^5 y^6 \quad \text{Ans.}$$

9. Middle term of $\left(\frac{a}{3} - \frac{b}{4}\right)^{12}$.
10. Middle terms of $\left(2x - \frac{y}{2}\right)^{15}$.
11. 6th term of $\left(a^{\frac{1}{3}} - b^{\frac{1}{2}}\right)^{16}$.

NOTE 3. Whether the index is positive or negative, integral or fractional, the Binomial Theorem can be applied equally well.

12. Expand $(a + b)^{-1}$.

$$(a+b)^{-1} = a^{-1} + (-1)\,a^{-2}b + (1)\,a^{-3}b^2 + (-1)\,a^{-4}b^3 +, \text{ etc.}$$
$$= a^{-1} - a^{-2}b + a^{-3}b^2 - a^{-4}b^3 +, \text{ etc.}$$
$$= \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \frac{b^3}{a^4} +, \text{ etc.}$$

13. Expand $(a - b)^{-1}$.

$$(a-b)^{-1} = a^{-1} - (-1)\,a^{-2}b + (1)\,a^{-3}b^2 - (-1)\,a^{-4}b^3 +, \text{ etc.}$$
$$= a^{-1} + a^{-2}b + a^{-3}b^2 + a^{-4}b^3 +, \text{ etc.}$$
$$= \frac{1}{a} + \frac{b}{a^2} + \frac{b^2}{a^3} + \frac{b^3}{a^4} +, \text{ etc.}$$

NOTE 4. The same results will be obtained in these examples if we write $(a + b)^{-1}$ and $(a - b)^{-1}$ in their other forms, $\frac{1}{a + b}$ and $\frac{1}{a - b}$, and perform the division.

Expand

14. $(m+n)^{-2}$ to five terms.

15. $(a-b)^{-3}$ to four terms.

16. $(1+3y)^{-1}$ to five terms.

17. $\left(\frac{a}{2}-x^2\right)^{-4}$ to five terms.

18. $\left(a-\frac{x}{4}\right)^{-7}$ to four terms.

19. Expand $(a+b)^{\frac{1}{2}}$.

$$(a+b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \left(\tfrac{1}{2}\right)a^{-\frac{1}{2}}b + \left(-\tfrac{1}{8}\right)a^{-\frac{3}{2}}b^2 + \left(\tfrac{1}{16}\right)a^{-\frac{5}{2}}b^3 +, \text{ etc.}$$
$$= a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}b - \tfrac{1}{8}a^{-\frac{3}{2}}b^2 + \tfrac{1}{16}a^{-\frac{5}{2}}b^3 -, \text{ etc.}$$
$$= \sqrt{a} + \frac{b}{2\sqrt{a}} - \frac{b^2}{8\sqrt{a^3}} + \frac{b^3}{16\sqrt{a^5}} -, \text{ etc.}$$

20. Expand $(a-b)^{\frac{1}{2}}$.

$$(a-b)^{\frac{1}{2}} = a^{\frac{1}{2}} - \left(\tfrac{1}{2}\right)a^{-\frac{1}{2}}b + \left(-\tfrac{1}{8}\right)a^{-\frac{3}{2}}b^2 - \left(\tfrac{1}{16}\right)a^{-\frac{5}{2}}b^3 +, \text{ etc.}$$
$$= a^{\frac{1}{2}} - \tfrac{1}{2}a^{-\frac{1}{2}}b - \tfrac{1}{8}a^{-\frac{3}{2}}b^2 - \tfrac{1}{16}a^{-\frac{5}{2}}b^3 -, \text{ etc.}$$
$$= \sqrt{a} - \frac{b}{2\sqrt{a}} - \frac{b^2}{8\sqrt{a^3}} - \frac{b^3}{16\sqrt{a^5}} -, \text{ etc.}$$

21. Expand $(a+b)^{-\frac{1}{2}}$.

$$(a+b)^{-\frac{1}{2}} = a^{-\frac{1}{2}} + \left(-\tfrac{1}{2}\right)a^{-\frac{3}{2}}b + \left(\tfrac{3}{8}\right)a^{-\frac{5}{2}}b^2 + \left(-\tfrac{5}{16}\right)a^{-\frac{7}{2}}b^3 +, \text{ etc.}$$
$$= a^{-\frac{1}{2}} - \tfrac{1}{2}a^{-\frac{3}{2}}b + \tfrac{3}{8}a^{-\frac{5}{2}}b^2 - \tfrac{5}{16}a^{-\frac{7}{2}}b^3 +, \text{ etc.}$$
$$= \frac{1}{\sqrt{a}} - \frac{b}{2\sqrt{a^3}} + \frac{3b^2}{8\sqrt{a^5}} - \frac{5b^3}{16\sqrt{a^7}} +, \text{ etc.}$$

22. Expand $(a-b)^{-\frac{1}{2}}$.

$$(a-b)^{-\frac{1}{2}} = a^{-\frac{1}{2}} - \left(-\tfrac{1}{2}\right)a^{-\frac{3}{2}}b + \left(\tfrac{3}{8}\right)a^{-\frac{5}{2}}b^2 - \left(-\tfrac{5}{16}\right)a^{-\frac{7}{2}}b^3 +, \text{ etc.}$$
$$= a^{-\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{3}{2}}b + \tfrac{3}{8}a^{-\frac{5}{2}}b^2 + \tfrac{5}{16}a^{-\frac{7}{2}}b^3 +, \text{ etc.}$$
$$= \frac{1}{\sqrt{a}} + \frac{b}{2\sqrt{a^3}} + \frac{3b^2}{8\sqrt{a^5}} + \frac{5b^3}{16\sqrt{a^7}} +, \text{ etc.}$$

Expand to five terms

23. $(m-n)^{\frac{1}{3}}$. 25. $(x-1)^{-\frac{1}{4}}$. 27. $(a+3a^2)^{\frac{1}{2}}$.

24. $(a-b)^{\frac{2}{3}}$. 26. $(1+3a)^{-\frac{2}{5}}$. 28. $(m-\frac{2}{3}n)^{\frac{3}{4}}$.

29. Find the 8th term in $(a+b)^{-3}$.

In the 8th term of $(a+b)^{-3}$ the index of a is -10, and the index of b is 7. The signs of all the terms (disregarding the sign of the coefficient) are $+$. In this example $n=-3$ and $r=8$. Hence, $n-r+2=-9$, and $r-1=7$; and the coefficient is

$$\frac{-3(-4)(-5)(-6)(-7)(-8)(-9)}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}=-36$$

Hence, the 8th term in $(a+b)^{-3}=-\dfrac{36\,b^7}{a^{10}}$.

30. Find the 7th term in $(x-y)^{-\frac{3}{4}}$.

The index of x in the 7th term will be $-\frac{27}{4}$, and the index of y is 6. The sign of the 7th term (disregarding the sign of the coefficient) is $+$. In this example $n=-\frac{3}{4}$, and $r=7$. Hence, $n-r+2=-\frac{23}{4}$, and $r-1=6$, and the coefficient is

$$\frac{-\frac{3}{4}\left(-\frac{7}{4}\right)\left(-\frac{11}{4}\right)\left(-\frac{\cancel{15}}{4}\right)\left(-\frac{19}{4}\right)\left(-\frac{23}{4}\right)}{1\cdot2\cdot\cancel{3}\cdot4\cdot\cancel{5}\cdot6}=\frac{100947}{12\times4^7}$$

Hence, the 7th term in $(x-y)^{-\frac{3}{4}}=\dfrac{100947\,y^6}{12\times4^7\sqrt[4]{x^{27}}}$.

Find the

31. 10th term in $(m-1)^{-2}$. 35. 7th term in $(m+n)^{-\frac{3}{4}}$.

32. 9th term in $(x+y^2)^{-5}$. 36. 9th term in $(a+1)^{-\frac{1}{5}}$.

33. 11th term in $(a-3)^{-4}$. 37. 6th term in $(2-b)^{-\frac{2}{3}}$.

34. 12th term in $(1+b)^{-6}$. 38. 8th term in $(a-b^2)^{-\frac{1}{3}}$.

39. Find $\sqrt{2}$; that is, expand $(1+1)^{\frac{1}{2}}$.

CHAPTER XXV.

LOGARITHMS.

373. Logarithms are exponents of the powers* of some number which is taken as a *base.* In the tables of logarithms in common use the number 10 is taken as the base, and all numbers are considered as powers of 10.

By Arts. 224, 231,

$10^0 = 1,$	that is, the logarithm of	1	is	0
$10^1 = 10,$	" " "	10	"	1
$10^2 = 100,$	" " "	100	"	2
$10^3 = 1000,$	" " "	1000	"	3
&c.,		&c.		

Therefore, the logarithm of any number between 1 and 10 is between 0 and 1, that is, is a fraction; the logarithm of any number between 10 and 100 is between 1 and 2, that is, is 1 plus a fraction; and the logarithm of any number between 100 and 1000 is 2 plus a fraction; and so on.

By Arts. 231, 232,

$10^0 = 1,$	that is, the logarithm of	1.	is	0
$10^{-1} = 0.1,$	" " "	0.1	"	-1
$10^{-2} = 0.01,$	" " "	0.01	"	-2
$10^{-3} = 0.001,$	" " "	0.001	"	-3
&c.,		&c.		

Therefore, the logarithm of any number between 1 and 0.1 is between 0 and -1, that is, is -1 plus a fraction; the logarithm of any number between 0.1 and 0.01 is between -1 and -2, that is, is -2 plus a fraction; and so on.

* The word *power* is used here to denote a number with any real index whatever.

The logarithm of a number, therefore, is either an integer (which may be 0) positive or negative, or an integer positive or negative and a fraction, which is always positive.

The representation of the logarithms of all numbers less than a unit by a *negative integer* and a *positive fraction* is merely a matter of convenience. The integral part of a logarithm is called the *characteristic,* and the decimal part the *mantissa.* Thus, the characteristic of the logarithm 3.1784 is 3, and the mantissa .1784.

374. The characteristic of the logarithm of a number is not given in the tables, but can be supplied by the following

Rule.

The characteristic of the logarithm of any number is equal to the number of places by which its first significant figure on the left is removed from units' place, the characteristic being positive when this figure is to the left, and negative when it is to the right of units' place.

Thus, the logarithm of 59 is 1 plus a fraction; that is, the characteristic of the logarithm of 59 is 1. The logarithm of 5417.7 is 3 plus a fraction; that is, the characteristic of the logarithm of 5417.7 is 3. The logarithm of 0.3 is -1 plus a fraction; that is, the characteristic of the logarithm of 0.3 is -1. The logarithm of 0.00017 is -4 plus a fraction; that is, the characteristic of the logarithm of 0.00017 is -4.

375. Since the base of this system of logarithms is 10, if any number is multiplied by 10, its logarithm will be increased by a unit (§ 72); if divided by 10, diminished by a unit (§ 78).

That is, the log of		5549.	being	3.7442
"	"	554.9	is	2.7442
"	"	55.49	"	1.7442
"	"	5.549	"	0.7442
"	"	0.5549	"	$\bar{1}.7442$
"	"	0.05549	"	$\bar{2}.7442$
"	"	0.005549	"	$\bar{3}.7442$

Hence, the mantissa of the logarithm of any set of figures is the same, wherever the decimal point may be.

As only the characteristic is negative, the minus sign is written *over* the characteristic.

TABLE OF LOGARITHMS.

376. To find the Logarithm of a Number of Two Figures.

Disregarding the decimal point, find the given number in the column **N** (pp. 373, 374), and directly opposite, in the column O, is the mantissa of the logarithm, to which must be prefixed the characteristic, according to the Rule in Art. 374.

Thus, the log of 85 is 1.9294
" " 26 " 1.4150

The first figure of the mantissa, remaining the same for several successive numbers, is not repeated, but left to be supplied.

Thus, the log of 83 is 1.9191

As, according to Art. 375, multiplying a number by 10 increases its logarithm by a unit, therefore, to find the logarithm of any number containing only two significant figures with one or more ciphers annexed, we use the same rule as above.

Thus, the log of 850 is 2.9294
" " 750000 " 5.8751

The principle just stated is applicable also in the cases that follow.

377. To find the Logarithm of a Number of Three Figures.

Disregarding the decimal point, find the first two figures in the column **N**, and the third figure at the top of one of the columns. Opposite the first two figures, and in the column under the third figure, will be the last three figures of the decimal part of the logarithm, to which the first figure in the column O is to be prefixed, and the characteristic, according to the Rule in Art. 374.

Thus, the log of 595 is 2.7745
" " 249 " 2.3962

In the columns **1, 2, 3**, etc., a small cipher ($_0$) or figure ($_1$) is sometimes placed below the first figure, to show that the figure which is to be prefixed from the column O has changed to the next larger number, and is to be found in the horizontal line directly below.

Thus, the log of 7960 is 3.9009
" " 25900 " 4.4133

378. To find the Logarithm of a Number of more than Three Figures.

On the right half of pages 373 and 374 are tables of Proportional Parts. The figures in any column of these tables are as many tenths of the average difference of the ten logarithms in the same horizontal line as is denoted by the number at the top of the column. The decimal point in these differences is placed as though the mantissas were integral.

1. To find the logarithm of a number of four figures, find as before the logarithm of the first three figures; to this, from the table of Proportional Parts, add the number standing on the same horizontal line and directly under the fourth figure of the given number.

Thus, to find the log of 5743.

	The log of 5740	is	3.7589
In Proportional Parts, in the same line, under	3	"	2.3
	Therefore, the log of 5743	"	3.7591

It is always best to find the logarithm of the nearest tabulated number, and add or subtract, as the case may be, the correction from the table of Proportional Parts.

Thus, to find the log of 6377.

$$6377 = 6380 - 3$$

The log of	6380	is	3.8048
correction for	3	"	2
Therefore, the log of	6377	"	3.8046

Whenever the fractional part omitted is larger than half the unit in the next place to the left, one is added to that figure.

2. For a fifth or sixth figure the correction is made in the same manner, only the point must be moved one place to the left for the fifth, two for the sixth figure.

Thus, to find the log of 3.6825.

The log of	3.68	is	0.5658
correction for	2	"	2.4
" "	5	"	.59
Therefore, the log of	3.6825	"	0.5661

To find the log of 112.82.

$$112.82 = 113 - 0.18$$

The log of	113	is	2.0531
correction for 0.18 is	(3.8 + 3.02)	"	6.82
Therefore, the log of	112.82	"	2.0524

379. To find the Number corresponding to a Given Logarithm.

Find in the table, if possible, the mantissa of the given logarithm. The two figures opposite in the column **N**, with the number at the head of the column in which the logarithm is found, affixed, and the decimal point so placed as to make the number of integral figures correspond to the characteristic of the given logarithm, as taught in Art. 374, will be the number required. Thus,

The number corresponding to log 5.5378 is 345000
" " " " 1.8745 " 74.9

If the mantissa of the logarithm cannot be exactly found, take the number corresponding to the mantissa nearest the given mantissa; in the same horizontal line in the table of Proportional Parts find the figures which express the difference between this and the given mantissa; at the top of the page, in the same vertical column, is the correction that belongs one place to the right of the number already taken, to be added if the given mantissa is greater, subtracted if less. Thus,

1. To find the number corresponding to

log	2.7660		
next less log,	2.7657,	and number corresponding,	583.
difference,	3	correction,	0.4
Number required,			583.4

2. To find the number corresponding to

log	$\bar{3}.8052$		
next greater log,	$\bar{3}.8055$,	and number corresponding,	0.00639
difference,	3	correction,	44
Number required,			0.0063856

The nearest number in the table of Proportional Parts to 3 is 2.7; corresponding to this at the top is 4, which belongs as a correction one place to the right of the number (0.00639) already taken, but

$3 - 2.7 = 0.3$; this, in like manner, gives a still further correction of 4, one place farther still to the right. The whole correction, therefore, is 44, to be deducted as shown above.

Find the logarithms of the following numbers:

3. 365. 4. 34700. 5. 83.24. 6. 0.00018.

Find the antilogarithms of the following numbers:

7. 2.095. 8. 1.346. 9. 3.62004. 10. $\bar{3}.83156$.

380. The great utility of logarithms in arithmetical operations is that addition takes the place of multiplication, and subtraction of division, multiplication of involution, and division of evolution. That is, to multiply numbers, we add their logarithms; to divide, we subtract the logarithm of the divisor from that of the dividend; to raise a number to any power, we multiply its logarithm by the exponent of that power; and to extract the root of any number, we divide its logarithm by the number expressing the root to be found.

This is the same as multiplication and division of different powers of the same letter by each other, and involving and evolving powers of a single letter; the number 10 takes the place of the given letter, and the logarithms are the exponents of 10.

MULTIPLICATION BY LOGARITHMS.

Rule.

381. *Add the logarithms of the factors, and the sum will be the logarithm of the product* (§ 72).

1. Multiply 246.5 by 0.003574. Ans. 0.881.

2. Find the product of 9.4478, 0.397526, 16.784. Ans. 63.06.

NOTE. It must be carefully borne in mind that the mantissa of the logarithm is *always* positive.

3. Multiply 0.00456 by 2.57.

	0.00456	log	$\bar{3}.6590$
	2.57	"	1.4082
Product,	0.1167	"	$\bar{1}.0672$

4. Multiply 36.75 by 0.003725.

Since the *numerical* product is the same whether the factors are positive or negative, we can use logarithms in multiplying when one or more of the factors are negative, taking care to prefix to the product the proper sign according to Art. 70. When a factor is negative, to the logarithm which is used n is appended.

5. Multiply -0.7546 by 0.00545.

	-0.7546	log	$\bar{1}.8777\,n$
	0.00545	"	$\bar{3}.7364$
Product,	-0.004113	"	$\bar{3}.6141\,n$

6. Find the product of -0.025, 625, and -12.125.

7. Find the product of -16, -67.23, and -0.008.
Ans. -8.606.

DIVISION BY LOGARITHMS.

Rule.

382. *From the logarithm of the dividend subtract the logarithm of the divisor, and the remainder will be the logarithm of the quotient* (§ 78).

1. Divide 34.56 by 0.0123.

	34.56	log	1.5386
	0.0123	"	$\bar{2}.0899$
Quotient,	2810	"	3.4487

2. Divide 18.5741 by 0.009496.

18.5741	log	1.2689
0.009496	"	$\bar{3}.9776$
Quotient, 1956	"	3.2913

Negative numbers can be divided in the same manner as positive, taking care to prefix to the quotient the proper sign, according to Art. 77.

3. Divide 84.52 by 3.514. Ans. 24.05.

4. Divide 5672 by 0.0037. Ans. 1533000.

5. Divide 0.053 by 797. Ans. 0.0000665.

6. Divide -16.54 by 345. Ans. -0.04794.

7. Divide -0.2456 by 25.05. Ans. -0.009806.

383. Instead of subtracting one logarithm from another, it is sometimes more convenient to add what it lacks of 10, and from the sum reject 10. The result is evidently the same. For

$$x - y = x + (10 - y) - 10$$

The remainder found by subtracting a logarithm from 10 is called the *arithmetical complement* of the logarithm, or the *cologarithm.* The cologarithm is easiest found by beginning at the left of the logarithm, and subtracting each figure from 9, except the last significant figure, which must be subtracted from 10.

By this method, Ex. 2 will appear as follows:

18.5741	log	1.2689
0.009496	colog	12.0224
Quotient, 1956	log	3.2913

DIVISION AND MULTIPLICATION BY LOGARITHMS.

384. In working examples combining multiplication and division, the use of cologarithms is of great advantage.

Rule.

Find the sum of the logarithms of the multipliers and the cologarithms of the divisors; reject as many tens as there are cologarithms (divisors); the result will be the logarithm of the number sought.

1. Find the value of $\dfrac{(39.74)\,(0.0861)\,(-470)}{(-684)\,(1.2475)}$.

39.74	log	1.5992
0.0861	"	$\bar{2}.9350$
-470	"	$2.6721\,n$
-684	colog	$7.1649\,n$
-1.2475	"	$9.9039\,n$
Ans. -1.8846	log	0.2751

An *even* number of negatives gives a positive result; an *odd* number, a negative (§ 70).

2. Find the value of $\dfrac{(42.5)\,(0.63)\,(-15)}{(0.725)\,(4.78)}$. Ans. -115.9.

3. Find the value of $\dfrac{(3.75)\,(-73)\,(0.056)}{(1.7498)\,(-125.13)}$. Ans. 0.0702.

PROPORTION BY LOGARITHMS.

Rule.

385. *Add the cologarithm of the first term to the logarithms of the second and third terms, and from the sum reject* 10.

1. Given $44 : 240 = 4522 : x$, to find x.

44	colog	8.3565
240	log	2.3802
4522	"	3.6553
Ans. 24662	"	4.3920

2. Given $324 : 672 = 125 : x$, to find x. Ans. 259.2.

3. Given $x : 9.426 = 908.4 : 15.42$, to find x. Ans. 555.3.

INVOLUTION BY LOGARITHMS.

Rule.

386. *Multiply the logarithm of the number by the exponent of the power required* (§ 206).

In involution, as the error in the logarithm is multiplied by the index of the power, the results with logarithms of only four decimal places cannot be relied on for more than three significant figures.

1. Find the 6th power of 2.34.

2.34	log	0.3692
		6
Ans. 164.1	"	2.2152

2. Find the 3d power of 0.000961.

0.000961	log	$\bar{4}.9827$
		3
Ans. 0.000000000748	"	$\overline{10}.9481$

3. Find the 6th power of 2.74119. Ans. 424.5.

4. Find the 4th power of 0.8724. Ans. 0.5791.

Negative numbers are involved in the same manner, taking care to prefix to the power the proper sign, according to Art. 205.

5. Find the 5th power of —0.225. Ans. —0.0005767.

6. Find the 8th power of —12.3. Ans. 523875000.

EVOLUTION BY LOGARITHMS.

Rule.

387. *Divide the logarithm of the number by the index of the root required* (§ 220).

When the characteristic is negative, and not divisible by the index of the root, we increase the negative characteristic so as to make it divisible, and to the mantissa prefix an equal positive number.

1. Find the 4th root of 0.254.

$$0.254 \qquad \log \quad \bar{2}.4048 = 4)\overline{\bar{4} + 2.4048}$$

$$\text{Ans. } 0.3992 \qquad \text{“} \quad \bar{1}.6012$$

2. Find the 3d root of 0.7589. Ans. 0.9121.

3. Find the 4th root of 0.0019. Ans. 0.2088.

Negative numbers are evolved in the same manner, taking care to prefix to the root the proper sign, according to Art. 213.

4. Find the 5th root of -0.037. Ans. -0.5172.

5. Find the 7th root of -0.000257. Ans. -0.307.

388. An **Exponential Equation**, that is, an equation having the unknown number as an exponent, may be solved by means of logarithms.

For, if $a^x = n$, by Art. 386,

$$x \quad \log a = \log n$$

$$\therefore \quad x = \frac{\log n}{\log a}$$

1. Solve $5^x = 625$.

$$x \cdot \log 5 = \log 625$$

$$\therefore \quad x = \frac{\log 625}{\log 5} = \frac{2.7959}{0.699} = 4 \quad \text{Ans.}$$

2. Solve $4913^x = 17$. 3. Solve $\left(\frac{1}{12167}\right)^x = 23$.

SYSTEMS OF LOGARITHMS.

389. The system of logarithms which has 10 for its base is the one in common use. It was first introduced in 1615, by Briggs, a contemporary of Napier, the inventor of logarithms. As in this system the mantissa of the logarithm of any set of figures is the same, wherever the decimal point may be (§ 375), which (in the Arabic notation of numbers) would not be the case with any other base, it is far the most convenient system. The number of possible systems, however, is infinite.

In general, if $a^x = n$, then x is the logarithm of n to the base a; and n is the number called the *antilogarithm* corresponding to the logarithm x, in a system whose base is a.

390. *The logarithm of* 1 *is* 0, *whatever the base may be.*

For the 0 power of every number is 1, or $a^0 = 1$ (§ 231).

391. *The logarithm of the base itself is* 1.

For the first power of any number is the number itself, or $a^1 = a$.

392. *Neither* 0 *nor* 1 *can be the base of a system of logarithms.*

For all the powers and roots of 0 are 0, and of 1 are 1.

393. *The logarithm of the reciprocal of any number is the negative of the logarithm of the number itself.*

For the reciprocal of any number is 1 divided by that number (§ 9); that is, it is the logarithm of 1 minus the logarithm of the number, or 0 minus the logarithm of the number (§ 382).

394. *In a system whose base is between* 1 *and* 0, *the less the number the greater its logarithm.*

For the greater the power of a proper fraction, the less its value. With such a base, the logarithms of numbers greater than 1 will be negative, less than 1 positive.

Thus, with $\frac{1}{3}$ as the base,

$$\begin{array}{llllll} \text{the log of } \frac{1}{9} & \text{is} & 2; & \text{of } \frac{1}{81} & \text{is} & 3 \\ \text{“} \quad \text{“} \quad 9 & \text{“} & -2; & \text{“} \quad 81 & \text{“} & -3 \end{array}$$

395. *The logarithms of numbers which form a geometric series form an arithmetic series.*

For, if a series increased or decreased by a constant ratio, its logarithms would increase or decrease by a constant difference equal to the logarithm of the constant ratio.

For an example see Art. 375; here the numbers decrease by the constant ratio 10, and the logarithms by the constant difference 1.

396. From the principles of the previous articles it will be easy to find the logarithms of the perfect roots and powers of any number. Thus,

In a system whose base is 8,

$$\begin{array}{lllr} 8^{\frac{1}{3}} = 2, & \text{that is, the log of} & & 2 = 0.\dot{3} \\ 8^{\frac{2}{3}} = 4, & \text{“} & \text{“} & 4 = 0.\dot{6} \\ 8^{1} = 8, & \text{“} & \text{“} & 8 = 1. \\ 8^{\frac{4}{3}} = 16, & \text{“} & \text{“} & 16 = 1.\dot{3} \\ \quad \&c., & & & \&c. \end{array}$$

Then, according to Art. 393,

$$\begin{array}{llll} \text{the log of} & \frac{1}{2} & = -0.\dot{3} & = \bar{1}.\dot{6} \\ \text{“} \quad \text{“} & \frac{1}{4} & = -0.\dot{6} & = \bar{1}.\dot{3} \\ \text{“} \quad \text{“} & \frac{1}{8} & = -1. & = \bar{1}. \\ \text{“} \quad \text{“} & \frac{1}{16} & = -1.\dot{3} & = \bar{2}.\dot{6} \\ & \&c., & & \&c. \end{array}$$

397. MISCELLANEOUS EXAMPLES.

1. In a system whose base is 3, what is the logarithm of 81? of 3? of 27? of 1? of $\frac{1}{3}$? of $\frac{1}{9}$? of 0?

2. Find the logarithms of $\frac{1}{25}$, $\frac{1}{625}$, 25, $\frac{1}{5}$, 5, to base $\frac{1}{5}$.

NOTE. $\text{Log}_4 256$ means the logarithm of 256 to base 4.

3. Find the value of

$$\log_6 \tfrac{1}{216},\quad \log_8 128,\quad \log_a \sqrt[3]{a^{-\frac{15}{2}}},\quad \log_a \frac{1}{a^{\frac{1}{2}}}.$$

Ans. -3, $\frac{7}{3}$, $-\frac{5}{2}$, $-\frac{1}{2}$.

4. Find the logarithm of 1000 to base 0.01, and of 0.0001 to base 0.001. Ans. $-\frac{3}{2}$, $\frac{4}{3}$.

5. The logarithm of 0.5 is $\bar{1}.\dot{6}$, what is the base? Ans. 8.

6. Simplify $\log \sqrt[4]{729 \sqrt[3]{9^{-1} \cdot 27^{-\frac{4}{3}}}}$. Ans. log 3.

7. Simplify $\log \frac{75}{16} - 2 \log \frac{5}{9} + \log \frac{32}{243}$. Ans. log. 2.

8. What logarithms would you need to find to reduce

$$\tfrac{1}{7}\sqrt{\tfrac{5}{8}} \times 0.012 \sqrt[8]{\tfrac{7}{11}}\ ?$$

Ans. logs of 7, 2, 3, 11.

9. Find the product of 37.2, 3.72, 0.000372, and 37200. Ans. 1914.5.

10. Find the number of digits in $3^{12} \times 2^8$. Ans. 9.

11. Find the number of digits in $(875)^{16}$. Ans. 48.

12. Solve $(1.2)^x = 1.1$. Ans. $x = 0.5227$.

13. Find $\log_e 16.345$, where $e = 2.71828$. Ans. $2.794-$.

Verify the following equations:

14. $\dfrac{(213)\,(7.655)}{(3145)\,(718)} = 0.000722.$

15. $\dfrac{(47)\,(0.653)\,(12\frac{5}{6})}{(3576)\,(1520)} = 0.00007247.$

16. $(\frac{9}{8})^{21} = 11.83.$

17. $\sqrt[5]{\frac{13}{16}} = 0.9592.$

18. $\sqrt[3]{\dfrac{3^2 \cdot 5^4}{\sqrt{2}}} = 15.84.$

19. $\sqrt[6]{(\frac{16}{17})^5 \cdot (\frac{1}{20})^9} = 0.01063.$

20. $\sqrt[3]{\left\{\dfrac{(294)\,(125)}{(42)\,(32)}\right\}^2} = 9.076.$

21. $\sqrt[11]{0.43\sqrt[10]{8\sqrt[7]{0.7}}} = 0.9434.$

22. Solve $(12.9)\,(7^{3x}) = \sqrt{\dfrac{1944}{2^x}}.$ Ans. $x = 0.1987.$

23. Simplify $\left\{\dfrac{(3.75)\,(7.3)\,(0.056)}{(1.7498)\,(125.13)}\right\}^{\frac{5}{3}}.$ Ans. 0.000256.

24. Given $a = 25$, $r = 5$, $l = 78125$, to find n. Ans. 6.

25. Given $a = \frac{1}{9}$, $r = 3$, $s = 364\frac{4}{9}$, to find n. Ans. 8.

26. What is the amount of \$1000 for 25 years at 5% compound interest? Ans. \$3388.

27. Find the amount at 4% interest of \$500 for 10 years, compounded semiannually. Ans. \$743.

28. In how many years will a sum of money be doubled at 5% compound interest? Ans. 14.2 years.

N	0	1	2	3	4	5	6	7	8	9	PROPORTIONAL PARTS.								
											1	2	3	4	5	6	7	8	9
10	0000	043	086	128	170	212	253	294	334	374	4.1	8.3	12.4	16.6	20.7	24.8	29.0	33.1	37.3
11	414	453	492	581	569	607	645	682	719	765	3.8	7.6	11.3	15.1	18.9	22.7	26.5	30.2	34.0
12	792	828	864	899	934	969	$_0$04	$_0$38	$_0$72	$_1$06	3.5	7.0	10.4	13.9	17.4	20.9	24.3	27.8	31.3
13	1139	173	206	239	271	303	336	367	399	430	3.2	6.4	9.7	12.9	16.1	19.3	22.5	25.7	29.0
14	461	492	523	553	584	614	644	673	703	732	3.0	6.0	9.0	12.0	15.0	18.0	21.0	24.0	27.0
15	1761	790	818	847	875	903	931	959	987	$_0$14	2.8	5.6	8.4	11.2	14.0	16.8	19.6	22.4	25.2
16	2041	068	095	122	148	175	201	227	263	279	2.6	5.3	7.9	10.5	13 2	15.8	18.4	21.1	23 7
17	304	330	355	380	405	430	455	480	504	529	2.5	5.0	7.4	9.9	12.4	14.9	17.4	19.9	22.3
18	553	577	601	625	648	672	695	718	742	765	2.3	4.7	7.0	9.4	11.7	14.1	16.4	18.8	21.1
19	788	810	833	856	878	900	923	945	967	989	2.2	4.5	6 7	8.9	11.1	13.4	15.6	17.8	20.0
20	3010	032	054	075	096	118	139	160	181	201	2.1	4.2	6.4	8.5	10.6	12.7	14.8	17.0	19.1
21	222	243	263	284	304	324	345	365	385	404	2.0	4.0	6.1	8.1	10.1	12.1	14.1	16.2	18.2
22	424	444	464	483	502	522	641	560	579	598	1.9	3.9	5.8	7 7	9.7	11.6	13.5	15.4	17.4
23	617	636	655	674	692	711	729	747	766	784	1.8	3.7	5 5	7.4	9.2	11.1	12.9	14.8	16.6
24	802	820	838	856	874	892	909	927	945	962	1.8	3.5	5.3	7.1	8.9	10.6	12.4	14.2	16.0
25	3979	997	$_0$14	$_0$31	$_0$48	$_0$65	$_0$82	$_0$99	$_1$16	$_1$33	1.7	3.4	5.1	6.8	8.5	10.2	11.9	13.6	15.3
26	4150	166	183	200	216	232	249	265	281	298	1.6	3.3	4.9	6.6	8.2	9.8	11.5	13.1	14.8
27	314	330	346	362	378	393	409	425	440	456	1.6	3.2	4.7	6.3	7.9	9.5	11.1	12 6	14 2
28	472	487	502	518	533	548	564	579	594	609	1.5	3.0	4.6	6.1	7.6	9.1	10.7	12.2	13.7
29	624	639	654	669	683	698	713	728	742	767	1.5	2.9	4.4	5.9	7.4	8.8	10.3	11.8	13.3
30	4771	786	800	814	829	843	857	871	886	900	1.4	2.8	4.3	5.7	7.1	8.5	10.0	11.4	12.8
31	914	928	942	955	969	983	997	$_0$11	$_0$24	$_0$38	1.4	2.8	4.1	5.5	6.9	8.3	9.7	11.0	12 4
32	5051	065	079	092	105	119	132	145	159	172	1.3	2.7	4.0	5.3	6.7	8.0	9.4	10.7	12.0
33	185	198	211	224	237	250	263	276	289	302	1.3	2.6	3.9	5.2	6.5	7.8	9.1	10.4	11.7
34	315	328	340	353	366	378	391	403	416	428	1.3	2.5	3 8	5.0	6.3	7.6	8.8	10.1	11.3
35	5441	453	465	478	490	502	514	527	539	551	1.2	2.4	3.7	4.9	6.1	7.3	8.6	9.8	11.0
36	563	575	587	599	611	623	635	647	658	670	1.2	2.4	3.6	4.8	5.9	7.1	8.3	9.5	10.7
37	682	694	705	717	729	740	752	763	776	786	1.2	2.3	3.5	4.6	5 8	6 9	8.1	9.3	10.4
38	798	809	821	832	843	855	866	877	888	899	1.1	2.3	3.4	4.5	5 6	6.8	7 9	9.0	10.2
39	911	922	933	944	955	966	977	988	999	$_0$10	1.1	2.2	3.3	4.4	5.5	6.6	7.7	8.8	9.9
40	6021	031	042	053	064	075	085	096	107	117	1.1	2.1	3.2	4.3	5.4	6.4	7.5	8.6	9.7
41	128	138	149	160	170	180	191	201	212	222	1.0	2.1	3.1	4.2	5.2	6.3	7.3	8.4	9.4
42	232	243	253	263	274	284	294	304	314	325	1.0	2.0	3.1	4.1	5.1	6.1	7.2	8.2	9.2
43	335	345	355	385	375	385	395	405	415	425	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8 0	9.0
44	435	444	454	464	474	484	493	503	513	522	1.0	2.0	2.9	3.9	4.9	5.9	6.8	7.8	8.8
45	6532	542	551	561	571	580	590	599	609	618	1.0	1.9	2.9	3.8	4.8	5.7	6.7	7.6	8.6
46	628	637	646	656	665	675	684	693	702	712	0.9	1.9	2.8	3.7	4 7	5.6	6.5	7 5	8.4
47	721	730	739	749	758	767	776	786	794	803	0.9	1.8	2 7	3.7	4 6	5.5	6.4	7.3	8.2
48	812	821	830	839	848	857	866	875	884	893	0.9	1.8	2 7	3.6	4.5	5.4	6.3	7.2	8.1
49	902	911	920	928	937	946	955	964	972	981	0.9	1.8	2.6	3.5	4.4	5 3	6.1	7.0	7.9
50	6990	998	$_0$07	$_0$16	$_0$24	$_0$33	$_0$42	$_0$50	$_0$59	$_0$67	0.9	1.7	2.6	3.4	4.3	5.2	6.0	6.9	7.7
51	7076	084	093	101	110	118	126	135	143	152	0.8	1.7	2.5	3.4	4.2	5.1	5.9	6.7	7 6
52	160	168	177	185	193	202	210	218	226	236	0.8	1.7	2.5	3.3	4.1	5.0	5.8	6.6	7.4
53	243	251	259	267	275	284	292	300	308	316	0.8	1.6	2.4	3.2	4.1	4.9	5.7	6.5	7.3
54	324	332	340	348	356	364	372	380	388	396	0.8	1.6	2.4	3.2	4.0	4.8	5 6	6.4	7.2

N	0	1	2	3	4	5	6	7	8	9	PROPORTIONAL PARTS.								
											1	2	3	4	5	6	7	8	9
55	7404	412	419	427	435	443	451	459	466	474	0.8	1.6	2.3	3.1	3.9	4.7	5.5	6.3	7.0
56	482	490	497	505	513	520	528	536	543	551	0.8	1.5	2.3	3.1	3.8	4.6	5.4	6.1	6.9
57	559	566	574	582	589	597	604	612	619	627	0.8	1.5	2.3	3.0	3.8	4.5	5.3	6.0	6.8
58	634	842	649	657	664	672	679	686	694	701	0.7	1.5	2.2	3.0	3.7	4.5	5.2	5.9	6.7
59	709	716	723	731	738	745	752	760	767	774	0.7	1.5	2.2	2.9	3.6	4.4	5.1	5.8	6.6
60	7782	789	798	803	810	818	825	832	839	846	0.7	1.4	2.2	2.9	3.6	4.3	5.0	5.7	6.5
61	853	860	868	875	882	889	896	903	910	917	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.4
62	924	931	938	945	952	959	966	973	980	987	0.7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3
63	993	$_0$00	$_0$07	$_0$14	$_0$21	$_0$28	$_0$35	$_0$41	$_0$48	$_0$65	0.7	1.4	2.1	2.7	3.4	4.1	4.8	5.5	6.2
64	8062	069	075	082	089	096	102	109	116	122	0.7	1.3	2.0	2.7	3.4	4.0	4.7	5.4	6.1
65	8129	136	142	149	156	162	169	176	182	189	0.7	1.3	2.0	2.7	3.3	4.0	4.6	5.3	6.0
66	195	202	209	215	222	228	235	241	248	254	0.7	1.3	2.0	2.6	3.3	3.9	4.6	5.2	5.9
67	261	267	274	280	287	293	299	306	312	319	0.6	1.3	1.9	2.6	3.2	3.9	4.5	5.1	5.8
68	325	331	338	344	351	357	363	370	376	382	0.6	1.3	1.9	2.5	3.2	3.8	4.4	5.1	5.7
69	388	395	401	407	414	420	426	432	439	445	0.6	1.2	1.9	2.5	3.1	3.7	4.4	5.0	5.6
70	8451	457	463	470	476	482	488	494	500	506	0.6	1.2	1.8	2.5	3.1	3.7	4.3	4.9	5.5
71	513	519	525	531	537	543	549	555	561	567	0.6	1.2	1.8	2.4	3.0	3.6	4.3	4.9	5.5
72	573	579	585	591	597	603	609	615	621	627	0.6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4
73	633	639	645	651	657	663	669	675	681	686	0.6	1.2	1.8	2.4	3.0	3.5	4.1	4.7	5.3
74	692	698	704	710	716	722	727	733	739	745	0.6	1.2	1.7	2.3	2.9	3.5	4.1	4.7	5.2
75	8751	756	762	768	774	779	785	791	797	802	0.6	1.2	1.7	2.3	2.9	3.5	4.0	4.6	5.2
76	808	814	820	825	831	837	842	848	854	869	0.6	1.1	1.7	2.3	2.8	3.4	4.0	4.5	5.1
77	865	871	876	882	887	893	899	904	910	915	0.6	1.1	1.7	2.2	2.8	3.4	3.9	4.5	5.0
78	921	927	932	938	943	949	954	960	965	971	0.6	1.1	1.7	2.2	2.8	3.3	3.9	4.4	5.0
79	976	982	987	993	998	$_0$04	$_0$09	$_0$15	$_0$20	$_0$25	0.5	1.1	1.6	2.2	2.7	3.3	3.8	4.4	4.9
80	9031	036	042	047	053	058	063	069	074	079	0.5	1.1	1.6	2.2	2.7	3.2	3.8	4.3	4.9
81	085	090	096	101	106	112	117	122	128	133	0.5	1.1	1.6	2.1	2.7	3.2	3.7	4.3	4.8
82	138	143	149	154	159	165	170	175	180	186	0.5	1.1	1.6	2.1	2.6	3.2	3.7	4.2	4.7
83	191	196	201	206	212	217	222	227	232	238	0.5	1.0	1.6	2.1	2.6	3.1	3.6	4.2	4.7
84	243	248	253	258	263	269	274	279	284	289	0.5	1.0	1.5	2.1	2.6	3.1	3.6	4.1	4.6
85	9294	299	304	309	315	320	325	330	335	340	0.5	1.0	1.5	2.0	2.5	3.0	3.6	4.1	4.6
86	345	350	355	360	365	370	375	380	385	390	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
87	395	400	405	410	415	420	425	430	435	440	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
88	445	450	455	460	465	469	474	479	484	489	0.5	1.0	1.5	2.0	2.5	2.9	3.4	3.9	4.4
89	494	499	504	509	513	518	523	528	533	538	0.5	1.0	1.5	1.9	2.4	2.9	3.4	3.9	4.4
90	9542	547	552	557	562	566	571	576	581	586	0.5	1.0	1.4	1.9	2.4	2.9	3.4	3.8	4.3
91	590	595	600	605	809	614	619	624	628	633	0.5	0.9	1.4	1.9	2.4	2.8	3.3	3.8	4.3
92	638	643	647	652	657	661	666	671	675	680	0.5	0.9	1.4	1.9	2.3	2.8	3.3	3.8	4.2
93	685	689	694	699	703	708	713	717	722	727	0.5	0.9	1.4	1.9	2.3	2.8	3.3	3.7	4.2
94	731	736	741	745	750	754	759	763	768	773	0.5	0.9	1.4	1.8	2.3	2.8	3.2	3.7	4.1
95	9777	782	786	791	795	800	805	809	814	818	0.5	0.9	1.4	1.8	2.3	2.7	3.2	3.6	4.1
96	823	827	832	836	841	845	850	654	859	863	0.5	0.9	1.4	1.8	2.3	2.7	3.2	3.6	4.1
97	868	872	677	881	886	890	894	899	903	908	0.4	0.9	1.3	1.8	2.2	2.7	3.1	3.6	4.0
98	912	917	921	926	930	934	939	943	948	952	0.4	0.9	1.3	1.8	2.2	2.6	3.1	3.5	4.0
99	956	961	965	969	974	978	983	967	991	996	0.4	0.9	1.3	1.7	2.2	2.6	3.1	3.5	3.9

EXAMINATION PAPERS FOR ADMISSION TO HARVARD COLLEGE.

SEPTEMBER, 1884.

(Time allowed, $1\frac{1}{2}$ hours.)

1. Solve the equations

$$2x^2 + 3xy - 3y^2 + 12 = 0,$$
$$3x + 5y + 1 = 0,$$

and state what values of x and y belong together.

2. Solve the equation $\dfrac{a-c}{x-a} - \dfrac{x-a}{a-c} = \dfrac{3b(x-c)}{(a-c)(x-a)}$, reducing the results to their simplest form.

3. Find the sixth term of the 19th power of $\left(\sqrt[3]{x^2} - \dfrac{y^3}{2x}\right)$, reducing the result to its simplest form.

4. Find the greatest common divisor of
$2x^4 - 3x^3 + 2x^2 - 2x - 3$ and $4x^4 + 3x^2 + 4x - 3$.

5. Solve the equation

$$\sqrt{7-x} + \sqrt{3x+10} + \sqrt{x+3} = 0.$$

6. A vessel is half full of a mixture of wine and water. If filled up with wine, the ratio of the quantity of wine to that of water is ten times what it would be if the vessel were filled up with water. Find the ratio of the original quantity of wine to that of water.

JUNE, 1885.

(Time allowed, $1\frac{1}{2}$ hours.)

1. Three students, A, B, and C, agree to work out a series of difficult problems, in preparation for an examination; and each student determines to solve a fixed number every day. A solves 9 problems per day, and finishes the series 4 days before B; B solves 2 more problems per day than C, and

finishes the series 6 days before C. Find the number of problems, and the number of days given to them by each student.

2. Solve the following equation, reducing the answers to their simplest form:

$$\frac{2}{1+3x}-\left(\frac{a(1+2x)}{b(1+3x)}-\frac{b(3x-1)}{a(2x+1)}\right)=0.$$

3. Solve the equation $\dfrac{\sqrt{3}}{\sqrt{2x-1}-\sqrt{x-2}}=\dfrac{1}{\sqrt{x-1}}$.

4. A certain whole number, composed of three digits, has the following properties: 10 times the middle digit exceeds the square of half the sum of the digits by 21; if 99 be added to the number, the order of the digits is inverted; and if the number be divided by 11, the quotient is a whole number of two digits, which are the same as the first and last digits of the original number. Find the number.

5. Given $\dfrac{x+6y}{7x-2y}=8$; find the value of $\dfrac{10x-3y}{2x-y}$.

6. Find the greatest common divisor of
$3x^4-x^3-2x^2+2x-8$ and $6x^3+13x^2+3x+20$.

7. Find the square root of
$4-12x+5x^2+26x^3-29x^4-10x^5+25x^6$.

SEPTEMBER, 1885.

(Time allowed, $1\frac{1}{2}$ hours.)

1. A certain manuscript is divided between A and B to be copied. At A's rate of work, he would copy the whole manuscript in 18 hours; B copies 9 pages per hour. A finishes his portion in as many hours as he copies pages per hour; B is occupied 2 hours more than A upon his portion. Find the number of pages in the manuscript, and the numbers of pages in the two portions.

2. Solve the following equation, reducing the answers to their simplest form:

$$\frac{\frac{1}{2}(x-a)}{b(x+a)} = \frac{1}{a} - 2\,\frac{b - \left(x - \frac{2b^2}{a}\right)}{(x+a)^2}.$$

3. Solve the equations

$$\frac{3\sqrt{x} + 2\sqrt{y}}{4\sqrt{x} - 2\sqrt{y}} = 6, \qquad \frac{x^2+1}{16} = \frac{y^2 - 64}{x^2};$$

finding all the values of x and y, and showing which values belong together.

4. Two casks, of which the capacities are in the ratio of a to b, are filled with mixtures of water and alcohol. If the ratio of water to alcohol is that of m to n in the first cask, and that of p to q in the second cask, what will be the ratio of water to alcohol in a mixture composed of the whole contents of the two casks? Reduce the answer to its simplest form.

What does the answer (in its simplest form) become, if $m = q = 0$? and what is the simplest statement of the question in this case?

5. Find the 10th term of $(x-y)^{27}$; of $\left(\frac{9a}{\sqrt{b}} - \frac{2b}{\sqrt{a}}\right)^{27}$.

The numerical coefficients are not to be computed, but expressed in terms of their prime factors; the literal parts are to be reduced to the simplest form.

Note. The above five questions constitute the paper; and all applicants are expected to do them if possible. The following question is not required, and is not necessary to make a perfect exercise; but it may be added, at the discretion of the student, and will be counted to improve the quality of an imperfect exercise.

6. Reduce to its lowest terms

$$\frac{6x^5 - 9x^4 + 11x^3 + 6x^2 - 10x}{4x^5 + 10x^4 + 10x^3 + 4x^2 + 60x}.$$

JUNE, 1886.

(Time allowed, $1\frac{1}{2}$ hours.)

1. A boat's crew, rowing at half their usual speed, row three miles down a certain river and back again, in the middle of the stream, accomplishing the whole distance in 2 hours and 40 minutes. When rowing at full speed, they go over the same course in 1 hour and 4 minutes. Find (in miles per hour) the rate of the crew when rowing at full speed, and the rate of the current.

(Notice *both* solutions of this problem.)

2. Solve the equation

$$3\sqrt{x^3+17}+\sqrt{x^3+1}+2\sqrt{5x^3+41}=0.$$

Substitute the answers, when found, in the equation, and show in what manner the equation is satisfied.

3. Solve the equations

$$x+\frac{4y+1}{x+2y}=2(y+1), \qquad x+3y+1=0.$$

4. Solve the equation

$$\frac{(a+2b)x}{a-2b}=\frac{a^2}{a-2b}-\frac{4b^2}{x};$$

and reduce the answers to their simplest form.

5. Find the greatest common divisor and the least common multiple of $4x^3-4x^2-5x+3$ and $10x^2-19x+6$.

6. Find the 6th and the 25th terms of the 29th power of $(x-y)$; reducing the numerical coefficients to their prime factors, and not performing the multiplications.

Find the 6th term of the 29th power of $\left(\frac{\sqrt[3]{a}}{b}-\frac{b^2}{2a}\right)$; reducing exponents to their simplest form, and combining similar factors.

SEPTEMBER, 1886.

(Time allowed, $1\frac{1}{2}$ hours.)

1. Solve the equation

$$\frac{\frac{1}{5}[2b(x+1)]^2}{4bx^3+5ax}-a\left(\frac{1}{x}-\frac{5ax-4b}{4bx^2+5a}\right)=0;$$

and reduce the answers to their simplest forms.

2. Solve the equation $x^{-3}-x^3=7(x^3+1)$.

3. A and B have 4800 circulars to stamp for the mail; and mean to do them in two days, 2400 each day. The first day, A, working alone, stamps 800 circulars, and then A and B together stamp the remaining 1600; the whole job occupying 3 hours. The second day, A works 3 hours, and B 1 hour; but they accomplish only $\frac{9}{10}$ of their task for that day. Find the number of circulars which each stamps per minute, and the length of time that B works on the first day.

4. Find the value of x from the proportion

$$\frac{5ac}{b^2}\sqrt[3]{ab^2}:\sqrt[4]{\frac{9c^3}{a^2}}=x:\frac{3a^2}{2}\sqrt{\frac{3c}{ab}};$$

and express the answer with the use of only one radical sign.

5. Given the three expressions

$$\begin{aligned}&2x^4+x^3-8x^2-x+6,\\&4x^4+12x^3-x^2-27x-18,\\&4x^4+4x^3-17x^2-9x+18;\end{aligned}$$

find the greatest common divisor and the least common multiple of the *first two* of these expressions; also those of *the whole group of three.*

JUNE, 1887.

(Time allowed, 1 hour.)

1. Solve the following equation:

$$\sqrt{x-3}+\sqrt{3x+4}+\sqrt{x+2}=0.$$

Find two answers, and verify the positive answer by showing that it satisfies the equation.

2. A broker sells certain railway shares for \$3240. A few days later, the price having fallen \$9 per share, he buys, for the same sum, 5 more shares than he had sold. Find the price and the number of shares transferred on each day.

3. Solve the following equation, finding four values of x:

$$x^4 + (2a^2 + 3ab - 2b^2)^2 = 5(a^2 + b^2)x^2.$$

4. Reduce the following expression to its simplest form as a single fraction:

$$\frac{\dfrac{1-x^3}{1+x^3} - \dfrac{1-x}{1+x}}{\dfrac{1+x^2}{1-x^2} + \dfrac{1+x}{1-x}}$$

SEPTEMBER, 1887.

(Time allowed, 1 hour.)

1. Solve the following equation, finding four values of x:

$$(x+a)(x-b) - \frac{a^2(x+a)}{x+b} - \frac{b^2(x-b)}{x-a} = \frac{3a^2b^2}{(x-a)(x+b)}.$$

2. At 6 o'clock on a certain morning, A and B set out on their bicycles from the same place, A going north and B south, to ride until $1\frac{1}{2}$ P.M. A moved constantly northwards at the rate of 6 miles per hour. B also moved always at a fixed rate; but, after a while, he turned back to join A. Four hours after he turned, B passed the point at which A was when B turned; and, at $1\frac{1}{2}$ P.M., when he stopped, he had reduced, by one half, the distance that was between them at the time of turning.

Find B's rate, the time at which he turned, the distance between A and B at that time, and the time at which B would have joined A if the ride had been continued at the same rates of speed. Find the answers for *both solutions.*

3. Find the sixth term of each of the following powers:

$$(x-y)^7; \quad \left(\frac{6a^2}{7b\sqrt{b}} - \frac{b}{\sqrt{3a}}\right)^7.$$

4. Reduce the following fraction to its lowest term:

$$\frac{6x^4 - 13x^3 + 3x^2 + 2x}{6x^4 - 9x^3 + 15x^2 - 27x - 9}.$$

JUNE, 1888.

(Time allowed, 1 hour.)

1. Reduce the following expression to its simplest form as a single fraction:

$$\frac{\frac{1-x^2}{1+y}\left(\frac{x}{1+x}-1\right)}{1-\left(\frac{1}{1-y}-\frac{x^2+y^2-x+y}{1-y^2}\right)}.$$

2. Solve the following equations, finding, and reducing to their simplest forms, two sets of values of x and y:

$$(x+3y) : (2x-y) = \left(\frac{1}{b}-\frac{7}{3a}\right) : \frac{2}{b},$$

$$x^2 = \tfrac{1}{2}(xy + 3ay + 18a^2).$$

What are the answers, when $a = 2$ and $b = -3$?

3. Two travellers, A and B, go from P to Q at uniform but unequal rates of speed. A sets out first, travelling on foot at the rate of 20 minutes for every mile. B follows, going 1 mile while A traverses the distance $\frac{PQ}{80}$. B overtakes and passes A, 8 miles from P; and when B reaches Q, he is 9 miles ahead of A. Find the distance PQ, and B's rate of speed in minutes to the mile.

(Obtain two solutions.)

4. Two men, working separately, can do a piece of work in x days and y days, respectively; find an expression for the time in which both can do it, working together.

A is 20 years old, and B is -2 years older; what is the age of B?

What are the values of x which satisfy the equation $x^2 = 3x$?

5. Write out $(x - y)^{11}$.

Find the square root of

$$4x^5 - 12x^5 + 5x^4 + 26x^3 - 29x^2 - 10x + 25.$$

SEPTEMBER, 1888.

(Time allowed, 1 hour.)

1. Reduce the following expression to its lowest terms as a single fraction:

$$\frac{\frac{2x}{3}}{\frac{1}{x} - \frac{2x^3 + 11x^2 - 43x - 24}{14x^3 - 31x^2 - 31x - 6}}$$

2. Solve the following equations, finding, and reducing to their simplest forms, two sets of values of x and y:

$$\frac{a}{y + 4b} = \frac{2b}{x - y},$$

$$\frac{1}{(b - a)x} - \left(\frac{3}{(a + b)y} - \frac{1}{a^2 - b^2}\right) = 0.$$

What are the values of x and y, if $a = 3$ and $b = -1$?

3. Tristram is ten years younger than Launcelot; and the product of the ages they attained in 1870 is 96. Find the ages they attain in 1888.

(Two solutions.)

4. A sum of $100 is put at compound interest at 4 per cent per annum for x years; find a formula for the amount.

5. Write out the first five terms and the last five terms of $(x - y)^{31}$.

Find and reduce to its simplest form the fifth term of

$$\left(a^3 b - \frac{3b^{-2}}{\sqrt{a^5}}\right)^{31}.$$

EXAMINATION PAPERS FOR ADMISSION TO YALE COLLEGE.

JUNE, 1885.

1. Given $\frac{5x+2}{3}-\left(3-\frac{3x-1}{2}\right)=\frac{3x+19}{2}-\left(\frac{x+1}{6}+3\right)$, to find x.

2. Multiply $\frac{b-y}{a^3+y^3}$, $\frac{ca+cy}{b^2-by}$, $\frac{b^6+y^6}{b^2+y^2}$, and $\frac{b}{c}$.

3. Multiply $x-\frac{1}{2}(1-\sqrt{-3})$ by $x-\frac{1}{2}(1+\sqrt{-3})$.

4. Divide $x^2y^{-\frac{4}{3}}-2+x^{-2}y^{\frac{4}{3}}$ by $x^{\frac{1}{2}}y^{-\frac{1}{3}}-x^{-\frac{1}{2}}y^{\frac{1}{3}}$.

5. Given $91x^2-2x=45$, to find both values of x.

6. Given $\frac{7}{\sqrt{x}}+\frac{4}{\sqrt{y}}=4$,

$\frac{1}{\sqrt{x}}+\frac{2}{\sqrt{y}}=1$, to find x and y.

7. Expand by the Binomial Theorem to five terms $(1+a)^{xy}$.

8. In Arithmetical Progression, given $d=$ the common difference, $a=$ the first term, and $s=$ the sum of series; derive the formula for $l=$ the last term.

9. If $\frac{\sqrt{a-bx}+\sqrt{c-mx}}{\sqrt{a-bx}+\sqrt{nx-d}}=\frac{\sqrt{a-bx}-\sqrt{c-mx}}{\sqrt{a-bx}-\sqrt{nx-d}}$, prove by using the principles of proportion that $\frac{c-mx}{nx-d}=1$.

SEPTEMBER, 1885.

1. Reduce $\dfrac{\dfrac{a^2}{b^3}+\dfrac{1}{a}}{\dfrac{a}{b}-\dfrac{1}{b}+\dfrac{1}{a}}$ to a simple fraction.

2. Find the greatest common divisor of $x^4 - 6x^2 - 8x - 3$ and $4x^3 - 12x - 8$.

3. Given $\sqrt{13+x} + \sqrt{13-x} = 6$, to find x.

4. Given $x^4 - 21x^2 = 100$, to find four values for x.

5. Find the value of $a^{\frac{1}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{1}{3}}$ when $a = 8$ and $b = 64$.

6. Given $\left\{\begin{array}{l} x + y = a \\ x^2 - y^2 = b^2 \end{array}\right\}$, to find x and y.

7. Given $(x^2 - ax) : \sqrt{x} :: \sqrt{x} : x$, to find values of x.

8. Expand $\dfrac{1}{(2a-3)^{\frac{1}{2}}}$ into a series.

9. Compute the value of the continued fraction

$$\cfrac{1}{12+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}}}$$

JUNE, 1886.

1. Divide $\dfrac{c-b}{c+b} - \dfrac{c^3-b^3}{c^3+b^3}$ by $\dfrac{c+b}{c-b} + \dfrac{c^2+b^2}{c^2-b^2}$.

2. Divide $x^2y^{-\frac{4}{3}} - 2 + x^{-2}y^{\frac{4}{3}}$ by $x^{\frac{1}{2}}y^{-\frac{1}{3}} - x^{-\frac{1}{2}}y^{\frac{1}{3}}$.

3. Multiply $\sqrt{-a} + c\sqrt[3]{b}$ by $\sqrt{-a} - c\sqrt[3]{b}$.

4. In $\dfrac{1}{\sqrt{3}-1}$ make the denominator rational, and compute the value of the expression to three places of decimals.

5. Given $a + x = \sqrt{a^2 + x\sqrt{b^2 + x^2}}$, to find x.

6. Solve the equations $\begin{cases} x + y = 12, \\ x^2 + y^2 = 74. \end{cases}$

7. If $A : B = C : D$, prove by the principles of proportion that $A^2 - B^2 : B^2 = C^2 - D^2 : D^2$.

8. Find the sum of the infinite series $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} +$ etc.

9. Expand to four terms by the Binomial Theorem $\dfrac{1}{\sqrt{1+x^2}}$.

SEPTEMBER, 1886.

1. Divide $\dfrac{x^4 - y^4}{x^2 - 2xy + y^2}$ by $\dfrac{x^2 + xy}{x - y}$.

2. Multiply $a^{\frac{5}{2}} - a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} - ab + a^{\frac{1}{2}} b^{\frac{4}{3}} - b^{\frac{5}{3}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{3}}$.

3. Free the fraction $\dfrac{1 - a^{-z} - y^2}{1 - x^{-3}y^{-2} + x^{-2}}$ from negative exponents.

4. Find x from $\dfrac{7x + 9}{4} - \left(x - \dfrac{2x - 1}{9}\right) = 7$.

5. Find x, y, and z from $\begin{cases} a = y + z, \\ b = x + z, \\ c = x + y. \end{cases}$

6. Multiply $x - 5 + 2\sqrt{-1}$ by $x - 5 - 2\sqrt{-1}$.

7. Make the denominator of the following fraction rational:

$$\frac{\sqrt{x}-\sqrt{x+y}}{\sqrt{x}+\sqrt{x+y}}.$$

8. Solve the equation $\frac{1}{x-1}+\frac{2}{x-2}=\frac{4}{3}$.

9. If $a:b=c:d$, prove by the principles of proportion that

$$\frac{a+b+c+d}{a+b-c-d}=\frac{a-b+c-d}{a-b-c+d}.$$

10. In a geometrical progression, having given first term, ratio, and sum of series, write formula for last term.

11. Expand to 4 terms $(a+x)^{-\frac{1}{4}}$.

June, 1887.

1. Resolve each of the following expressions into three factors:

$$a^4 b + 8\,a\,c^8\,b\,m^6, \qquad 4\,c^3 x^2 + 4\,c^2 x\,y + c\,y^2.$$

2. Divide $\frac{a}{a-b}-\frac{b}{a+b}$ by $\frac{b}{a-b}-\frac{a}{a+b}$.

3. Multiply $\left(x-\frac{1-\sqrt{3}}{2\sqrt{2}}\right)\left(x-\frac{1+\sqrt{3}}{2\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)$.

4. Solve $\sqrt{x+40}=10-\sqrt{x}$.

5. Solve $m\,x^2+m\,n=2\,m\sqrt{n\,x}+n\,x^2$.

6. Given $\frac{15}{x}:\frac{21}{y}::3:7$, and $x^2-y^2=9$, to find x and y.

7. Expand by the Binomial Theorem $3\,b\,(2\,x-y)^{\frac{1}{2}}$.

JUNE, 1888.

1. Remove the parentheses from the following expression and reduce it to its simplest form.

$$5x - (3x - 4) - [7x + (2 - 9x)].$$

2. Resolve each of the following expressions into as many factors as possible.

(*a*.) $x^6 - 1.$

(*b*.) $(x^2 + y^2 - z^2)^2 - 4x^2y^2.$

3. Divide $\frac{1}{1-x} - \frac{1}{1+x}$ by $\frac{1}{1-x} + \frac{1}{1+x}$.

4. Solve the equations

$$\frac{3}{x} + \frac{1}{y} = \frac{5}{4}.$$

$$\frac{2}{x} - \frac{3}{y} = -1.$$

5. Solve the equation $\sqrt{x-3} - \sqrt{2x+8} = -3.$

6. Solve the equation $x^3 - x^{\frac{3}{2}} = 256.$

7. Multiply $x + 3 - 2\sqrt{-1}$ by $x + 3 + 2\sqrt{-1}.$

8. Expand $(x^3 + b)^{-\frac{1}{3}}$ to four terms.

9. Given the series $y = x - \frac{x^2}{2} + \frac{x^3}{4} - \frac{x^4}{8} +$ etc., to find the value of x in terms of y.

EXAMINATION PAPERS FOR ADMISSION TO AMHERST COLLEGE.

SEPTEMBER, 1884.

1. Find the greatest common divisor of $x^3 - 2x^2 - x + 2$ and $x^4 - 3x^3 + 3x^2 - 3x + 2$.

2. The sum of the digits of a number of two figures is 9; and if 9 be subtracted from the number the digits are reversed. What is the number?

3. $\left.\begin{aligned} ax + by &= c \\ a'x + b'y &= c' \end{aligned}\right\}$; find x and y.

4. The sum of the squares of two consecutive numbers is 545. What are the numbers?

5. Solve $\left\{\begin{aligned} x - y &= 2. \\ x^2 - y^2 &= 20. \end{aligned}\right.$

6. Raise $2\sqrt[6]{3x^5y}$ to the 4th power and reduce the result to its simplest form.

7. What number added to 2, 20, 9, 34, will make the result proportional?

8. Given d, the difference, n, the number of terms, and s, the sum of an arithmetical progression; find the formula for l, the last term.

9. Find the sum of the infinite series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} +$ etc.

10. Find the 4th term of $(x - 2y)^{12}$ by the Binomial Theorem.

JUNE, 1885.

1. Divide $\dfrac{x-y}{x^2+2xy+y^2}$ by $\dfrac{x^2-y}{x+y}$.

2. Factor $x^4-2x^2y^2+y^4$.

3. Reduce the product $(3x^{-2}y^3z^{-4})(5x^3y^{-4}z^3)$ to its simplest form, freed from negative exponents.

4. Solve $\sqrt{x+13}=1+\sqrt{x}$.

5. Solve $x^2+4ax=b$.

6. Given $\begin{cases} x^2-xy=15, \\ x^2-y^2=21; \end{cases}$ find x and y.

7. Given $\dfrac{3x}{4}+\dfrac{x-4}{2}-\dfrac{x-10}{2}=x-6$, to find x.

8. What is the sum of the first twenty odd numbers?

9. The first term of a geometrical progression is $\frac{1}{4}$, the ratio $\frac{1}{2}$, the number of terms 7. Find the sum.

10. Give the fourth term of $(2x-3y)^{-3}$.

SEPTEMBER, 1885.

1. From $a-\dfrac{b-c}{2}$ take $\dfrac{a-b}{3}-x$.

2. Divide a^6-1 into its prime factors.

3. Reduce to its lowest terms $\dfrac{4x^3-5x^2+x}{8x^2-6x+1}$.

4. Solve $\begin{aligned} 3x-4y&=-6, \\ 10x+2y&=26. \end{aligned}$

5. Solve $2x^2-5x+2=0$.

6. Solve $\begin{aligned} x^2 + y^2 &= 34, \\ x + y &= 8. \end{aligned}$

7. What two numbers whose difference is d are to each other as a to b.

8. Insert 5 arithmetical means between 2 and -3.

9. What is the sum of the first 10 terms of the series 1, 2, 4, 8, etc.?

10. Expand $(x^2 - 2b)^6$.

JUNE, 1886.

1. From $\dfrac{a + b - 2c}{3}$ take $\dfrac{2a - b + c}{4}$.

2. Reduce $\dfrac{3x^2y + 3xy^2}{3x^2 + 6xy + 3y^2}$ to its lowest terms.

3. Given $3x - \dfrac{x - 4}{4} - 4 = \dfrac{5x + 14}{3} - \dfrac{1}{12}$, to find x.

4. Solve $x + \sqrt{x + 3} = 4x - 1$.

5. Given $2x - \dfrac{y + 3}{4} = 8,\ 4y - \dfrac{8 - x}{3} = 24\frac{1}{2} - \dfrac{2y + 1}{2}$; find x and y.

6. Find $\sqrt{24} + \sqrt{54} - \sqrt{6}$.

7. A person sets out from a certain place, and goes at the rate of 11 miles in 5 hours; and, 8 hours after, another person sets out from the same place, and goes after him at the rate of 13 miles in 3 hours. How far must the latter travel to overtake the former?

8. The 1st and 9th terms of an arithmetical progression are 5 and 22. Find the sum of 21 terms.

9. Find the 12th term of the geometrical progression

$$\sqrt{2},\ -2,\ +2\sqrt{2},\ -4, \text{ etc.}$$

10. Find the first four terms of $(2x - 3y)^5$ by the Binomial Formula.

SEPTEMBER, 1886.

1. Reduce $(a+b-c)^2 + (a-b+c)^2$ to its simplest form.

2. Reduce $\dfrac{2x^3 - 16x - 6}{3x^3 - 24x - 9}$ to its lowest terms.

3. Given $\dfrac{3x+4}{5} - \dfrac{7x-3}{2} = \dfrac{x-16}{4}$; find x.

4. Find x and y from the equations

$$\frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4},$$
$$\frac{2y+4}{3} - \frac{2x+y}{8} = \frac{x+13}{4}.$$

5. Solve $2x - \sqrt{2x-1} = x + 2$.

6. Solve $\begin{cases} x^2 + y^2 = 50, \\ 9x + 7y = 70. \end{cases}$

7. Reduce $\sqrt{45c^3} - \sqrt{80c^3} + \sqrt{5a^2c}$ to its simplest form.

8. Find the sum of the first 90 odd numbers by arithmetical progression.

9. Find the sum of the geometrical progression 20, 19, $18\frac{1}{20}$, etc.

10. Find the first four terms of $(1-x)^{12}$ by the Binomial Formula.

JUNE, 1887.

1. Reduce $(a + b - c)\sqrt{x + y} - (a + b + c)(x + y)^{\frac{1}{2}}$ to its simplest form.

2. Resolve $a^6 - b^6$ into its prime factors.

3. Divide $\dfrac{2\,a^2\,x^{\frac{4}{3}}\,y}{6\,b^3\,c^{\frac{1}{4}}\,d^2}$ by $\dfrac{a\,x^{\frac{1}{3}}\,y^2}{b^2\,c^{\frac{1}{4}}\,d^2\,e}$

4. $\dfrac{3\,x + 2\,a}{2} - \dfrac{x - 5\,a}{3} = 5\,a$; find x.

5. What number multiplied by m gives a product a less than n times the number?

6. $\begin{matrix} 5\,x + 3\,y = 19 \\ 7\,x - 2\,y = 18 \end{matrix}$; find x and y.

7. Find the square root of $a + 2\,a^{\frac{1}{2}}\,x^{\frac{1}{2}} + x$.

8. Find the square root of $81\,a^4\,x^{-2}\,y^{\frac{2}{3}}\,z^{-\frac{1}{5}}$.

9. Find the roots of $a\,x^2 + b\,x + c = 0$.

10. $x^2 + xy = 10$, $xy - y^2 = -3$; find x and y.

SEPTEMBER, 1887.

1. Reduce $a - [2\,b - (3\,c + 2\,b - a)]$ to its simplest form.

2. Divide $a^n\,b^{m-n}$ by $a^{n-m}\,d^{-n}$.

3. Reduce $\dfrac{\dfrac{c}{c-1} - 1}{1 - \dfrac{c}{c+1}}$ to its simplest form.

4. Given $\left\{\begin{matrix} x + 2\,y = 7 \\ 2\,x + 3\,y = 12 \end{matrix}\right\}$, to find x and y.

5. Reduce $a\sqrt{48\,a^3\,d}$ and $\sqrt{\frac{45}{49}}$ to simpler forms.

6. Multiply $3\sqrt{\dfrac{a}{3}}$ by $2\sqrt{\dfrac{a}{6}}$.

7. Given $\dfrac{x-5}{3}+\dfrac{x}{2}=12-\dfrac{x-10}{3}$, to find x.

8. Find two numbers whose sum equals s, and whose difference equals d.

9. Solve the equation $3\,x^2-4\,x=119$.

10. Find the first four terms of $(x-2\,y)^7$.

JUNE, 1888.

1. Resolve $16\,a^6\,b^4\,m^2-8\,a^3\,b^2\,m+1$ into its factors.

2. Find the greatest common divisor of $6\,x^3-6\,x^2\,y+2\,x\,y^2-2\,y^3$ and $12\,x^2-15\,x\,y+3\,y^2$.

3. $\dfrac{x+3}{2}-\dfrac{x-2}{3}=\dfrac{3\,x-5}{12}+\dfrac{1}{4}$; find x.

4. $\dfrac{x+y}{2}-\dfrac{x-y}{3}=8$; $\dfrac{x+y}{3}+\dfrac{x-y}{4}=11$; find x and y.

5. $x-2y+3z=2$; $2x-3y+z=1$; $3x-y+2z=9$; find x and y.

6. Solve the equation $\sqrt{x+5}=\dfrac{12}{\sqrt{x+12}}$.

7. Solve the equations $x+y=a$; $x^2+y^2=b^2$.

8. Find the sum $\frac{3}{5}\sqrt{\frac{2}{3}}$ and $\frac{3}{4}\sqrt{\frac{25}{6}}$.

9. Demonstrate the fundamental formulæ used in Arithmetical Progression. Find the sum of the first n terms of the progression 1, 3, 5, 7, etc.

10. Find the sum of the first n terms of a geometrical progression whose first term is a, and third term c.

SEPTEMBER, 1888.

1. Find the value of $\dfrac{x+2a}{x-2a}+\dfrac{x+2b}{x-2b}$ when $x=\dfrac{4ab}{a+b}$.

2. Resolve $1-c^4$ into its prime factors.

3. Multiply together $\dfrac{1-x^2}{1+y}$, $\dfrac{1-y^2}{x+x^2}$, and $1+\dfrac{x}{1-x}$.

4. Solve the equation $\dfrac{x+3}{x-3}-\dfrac{x-3}{x+3}=a$.

5. $\dfrac{x}{5}+\dfrac{y}{6}=18$; $\dfrac{x}{2}-\dfrac{y}{4}=21$; find x and y.

6. $x^2+y^2=34$; $xy=15$; find x and y.

7. Reduce $\dfrac{x}{a-\sqrt{a^2-x^2}}$ to an equivalent fraction having a rational denominator.

8 Find the ratio of an infinite decreasing geometrical progression of which the first term is 1, and the sum of the terms is $\frac{4}{3}$.

9. Find the sum of the terms of an arithmetical progression formed by inserting 9 arithmetical means between 9 and 109.

10. Expand $(a-b)^5$ by the Binomial Formula.

EXAMINATION PAPERS FOR ADMISSION TO DARTMOUTH COLLEGE.

1885.

1. Factor $x^4 - 16a^4$, $x^6 + a^6$, $x^2 - 2ax - b^2 + a^2$, $x^2 - x - 72$.

2. Find the greatest common divisor and least common multiple of $x^3 - 3x - 2$ and $x^5 - 2x^4 - x + 2$.

3. Simplify $\dfrac{1 - \dfrac{2 + \frac{1}{x}}{2 - \frac{1}{x}}}{1 - \dfrac{x + \frac{1}{2}}{x - \frac{1}{2}}}$ and $\dfrac{8}{\sqrt{-3} - 1} - (1 - \sqrt{-3})^2$.

4. Solve $x - \left(\dfrac{1-x}{4} - \dfrac{1+x}{2}\right) = \dfrac{x+3}{2}$.

5. Solve $\begin{cases} \dfrac{x}{2} + 2y = x - y + 4. \\ 2x - 3y = y - x + 4. \end{cases}$

6 Solve $\sqrt{x+2} - \sqrt{x-2} = \sqrt{2x}$.

7. Find the value of $9^{\frac{1}{2}} \times 8^{-\frac{4}{3}} \times 7^0 \times 4^{-\frac{1}{2}} \div (8^{-2} \times 3)^{-1}$.

8. Multiply $x - x^{-1}$ by $x - x^{-1}$, $2 + \sqrt{-3}$ by $2\sqrt{3}$, and $x^{\frac{1}{3}} - y^{\frac{2}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.

1886.

1. Factor $x^2 - 9a^4$, $x^9 + y^9$, $x^2 + 4xy - 4 + 4y^2$, $x^2 + 3ax + 2a^2$.

2. Reduce $\left\{\dfrac{\dfrac{x^6+1}{x^2+1}}{\dfrac{x^3+1}{x+1}}\right\}[1 + x(x-1)]$ to its simplest form.

3. Multiply $x + x^{-1} - 1$ by $x - x^{-1} + 1$, and $x^{\frac{1}{2}} + y^{\frac{1}{2}}$ by $x^{\frac{3}{2}} - x y^{\frac{1}{2}} + x^{\frac{1}{2}} y - y^{\frac{3}{2}}$.

4. Divide $x - y$ by $x^{\frac{1}{4}} + y^{\frac{1}{4}}$, and $x^3 + x^{-3}$ by $x + x^{-1}$.

5. Write the value of $8^{\frac{1}{3}} \times 9^{-\frac{1}{2}} \times 2^{-1} \times 3^0 \times 1^{-2} \times \sqrt[4]{81}$.

6. Solve $x - 3 - \left(\frac{x-4}{4} - \frac{x-6}{7}\right) = \frac{x}{\frac{12}{5}}$.

7. Solve $\begin{cases} \frac{x}{4} - \frac{y}{3} = x - 4 \\ \frac{x}{8} - y = y - 5\frac{1}{2} \end{cases}$ and $\begin{cases} \frac{1}{x} + \frac{1}{y} = 5. \\ \frac{2}{x} + \frac{3}{y} = 13. \end{cases}$

1887.

1. Give all the theorems used in factoring binomials.

2. Find the prime factors of $1 + a^6$, $a^6 - b^3$, $a^4 + 4b^4 + 4a^2b^2 - c^3$, $x^2 - x - 20$, $x^3 + x^2 - 8x - 12$.

3. Find the G. C. D. of $x^4 + 4x^3 + 12x^2 + 16x + 16$ and $4x^3 + 12x^2 + 24x + 16$.

4. Solve $\frac{1}{y} + \frac{1}{x} = 7$, $\frac{2}{y} - \frac{3}{x} = 2$.

5. Write the values of $27^{-\frac{2}{3}}$, $27^{\frac{4}{3}}$, 27^0, $\left[(27^{-2})^{-\frac{1}{3}}\right]^{-\frac{1}{2}}$.

6. Reduce to simplest form

$$\frac{a^{-1} - b^{-1}}{(a-b)^{-1}} \times ab, \qquad \sqrt{x^2 + \frac{1}{x^2} - 2},$$

$$(11 + 4\sqrt{6})^{\frac{1}{2}}, \qquad (-1 - \sqrt{-3})^3.$$

7. Reduce to equivalent fractions having rational denominators

$$\frac{ac}{3a^{\frac{1}{2}}b^{\frac{3}{4}}c^{-\frac{1}{3}}}, \qquad \frac{1}{a^{\frac{2}{3}} + b^{\frac{1}{2}}}.$$

8. Solve $\sqrt{x+5} + \sqrt{x-8} = \sqrt{3}$.

1888.

1. Remove the parentheses from

$$3a - \{3a - [3a - (3a - \overline{3a - 3a}) - 3a] - 3a\} - 3a,$$

and simplify the result.

2. Give the three theorems used in factoring binomials.

3. Factor $4a^2x^4 - 9b^4c^8$, $8b^2c^3 + 48b^2c^2 + 72b^2c$, $x^4 - 3x^3 - 14x^2 + 48x - 32$.

4. Resolve $1 - a^8$ into six factors.

5. Give two methods of finding the G. C. D. of two quantities.

6. Reduce $\dfrac{\dfrac{a^2+b^2}{b} - a}{\dfrac{1}{a} - \dfrac{1}{b}} \times \dfrac{a^2 - b^2}{a^3 + b^3}$ to a simple fraction and lowest terms.

7. Solve $\begin{cases} 2x^{-1} - y^{-1} = 22x^0, \\ 3x^{-1} - 3y^{-1} = 18y^0. \end{cases}$

8. A and B can do $\frac{1}{4}$ of a piece of work in 2 days. B can do $\frac{1}{3}$ of it in 6 days. How long would it take A to do $\frac{1}{3}$ of it?

9. Reduce to simplest form

$$\sqrt{\frac{a^2+b^2}{4}},\ \sqrt{6+\sqrt{-13}} \times \sqrt{6-\sqrt{-13}},\ (35 - 12\sqrt{6})^{\frac{1}{2}}.$$

10. Reduce to equivalent fractions having rational denominators

$$\frac{ac}{c^{\frac{4}{3}}\left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)}, \qquad \frac{1}{a^{\frac{2}{3}} + b^{\frac{3}{4}}}.$$

11. Solve $\dfrac{1}{x + \sqrt{x^2 - 1}} + \dfrac{1}{x - \sqrt{x^2 - 1}} = 12.$

EXAMINATION PAPERS FOR ADMISSION TO BROWN UNIVERSITY.

JUNE, 1885.

1. Find highest common divisor of $15\,a^2x^3 - 20\,a^2x^2 - 65\,a^2x - 30\,a^2$ and $12\,bx^3 + 20\,bx^2 - 16\,bx - 16\,b$.

2. $\dfrac{k}{z} + \dfrac{z}{k} + \dfrac{k\,(z-k)}{z\,(z+k)} - \dfrac{z\,(z+k)}{k\,(z-k)} = \dfrac{kz}{k^2 - z^2} - 2.$

3. A number is compounded of three figures whose sum is 17. The figure of the hundreds is double that of the units. When 396 is subtracted the order of the figures is reversed. What is the number?

4. Multiply $2\sqrt{-3} - 3\sqrt{-2}$ and $4\sqrt{-3} + 6\sqrt{-2}$.

5. Reduce to an equivalent fraction with a rational denominator

$$\frac{\sqrt{x} - 4\sqrt{x-2}}{2\sqrt{x} + 3\sqrt{x-2}}.$$

6. $\sqrt{2x-3} - \sqrt{8x+1} + \sqrt{18x-92} = 0$. Find value of x.

7. $2x^2 - 2xy - y^2 = 3$; $x^2 + 3xy + y^2 = 11$. Find values of x and y.

8. In an arithmetical progression, given the last term, -47; the common difference, -1; and the sum of the terms, -1118; find the first term and the number of terms.

9. In a geometrical progression, given the first term, $\frac{2}{3}$; the ratio, $-\frac{1}{2}$; and the number of terms, 7; find the sum of the terms.

10. Develop by Binomial Formula $(a^2b - \frac{1}{3}\,x\,a^{-2})^4$.

SEPTEMBER, 1885.

1. Find the least common multiple of $2x^3 - 3x^2 - x + 1$ and $6x^3 - x^2 + 3x - 2$.

2. $\frac{x}{2} - \frac{a - bcx}{2bc} = \frac{x}{6c} - \frac{ac - 4bx}{3bc}$. Find value of x.

3. A number is compounded of three figures whose sum is 17. The figure of the units is two thirds that of the hundreds. When 297 is subtracted the order of the figures is reversed. What is the number?

4. Multiply $3\sqrt{-1} - 2\sqrt{-2}$ and $4\sqrt{-2} - 2\sqrt{-1}$.

5. Reduce to an equivalent fraction with a rational denominator

$$\frac{\sqrt{a^2 - 1} - \sqrt{a^2 + 1}}{\sqrt{a^2 - 1} + \sqrt{a^2 + 1}}.$$

6. $\sqrt{3x} + \sqrt{3x + 13} = \frac{91}{\sqrt{3x + 13}}$. Find value of x.

7. $(2x - 5)^2 - (2x - 1)^2 = 8x - 5x^2 - 5$.

8. In an arithmetical progression, given the first term, $-\frac{2}{3}$; the number of terms, 18; and the last term, 5; find the common difference and sum of terms.

9. In a geometrical progression, given last term, -12; sum of terms, -255; and ratio, 2; find first term and number of terms.

10. Develop by Binomial Formula $(\frac{1}{2}ab^2 - \frac{2}{3}a^2b^{-1})^5$.

JUNE, 1886.

1. Multiply $5x^{p-3}y^{r+3} - 2x^{p-1}y^{r+1} - x^{p-2}y^{r+2}$ by $3x^{p+4}y^{r-1} + 4x^{p+5}y^{r-2} - x^{p+3}y^r$.

2. Simplify $\left[\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c}}} \div \dfrac{1}{a+\dfrac{1}{b}}\right] - \dfrac{1}{b[abc+a+c]}$.

3. Given $\begin{matrix} mx + 2ny = p; \\ 2sx + ty = q. \end{matrix}$ Find the values of x and y.

4. The smaller of two numbers divided by the larger is .21 with a remainder of .04162. The greater divided by the smaller is 4 with .742 for a remainder. What are these numbers?

5. Given $\dfrac{x(2x-10)}{12} - \dfrac{(x-7)^2}{2} = \dfrac{(14-x)^2}{3} + (11-x)^2$, to find value of x.

6. $\dfrac{2}{x+\sqrt{2-x^2}} + \dfrac{2}{x-\sqrt{2-x^2}} = x.$

7. Expand $(2x^{\frac{2}{3}} + 3x^{\frac{1}{3}}y)^4$.

8. Find sum of terms in a geometrical progression.

September, 1886.

1. $x - \left\{\dfrac{2y-3}{4} + \dfrac{3x-5}{6}\right\} = 2 - \dfrac{2x-3y-1}{12};$

$y + \left\{\dfrac{2x-y}{4} - \dfrac{3y-2}{3}\right\} = 1\frac{1}{6} - \dfrac{3-2x}{6}.$

Find values of x and y.

2. $\sqrt{a+x} + \sqrt{a-x} = \sqrt{6}$. Find values of x.

3. A number is compounded of three figures whose sum is 17. The figure of the hundreds is double that of the units. When 396 is subtracted the order of the figures is reversed. What is the number?

4. $3x^2 - 4x - 4 = 0$. Find values of x.

5. Find sum of six terms of the geometrical progression of which $\frac{8}{5}$ is the first term and $\frac{8}{3}$ the second term.

JUNE, 1887.

1. $5y - \dfrac{3x+7}{2} = 13\frac{1}{2}$,

$$\frac{4x-3}{3} - \frac{2x+3y}{4} = -5\tfrac{3}{8}.$$ Find values of x and y.

2. $2x - 3y = 8$.
$y - 3z = -11$.
$x - 2y + 4z = 17$. Find values of x, y, and z.

3. A boy spent his money in oranges. If he had bought 5 more, each orange would have cost a half-cent less; if 3 less, a half-cent more. How much did he spend, and how many did he buy?

4. Multiply $\sqrt{p+q} + \sqrt{p-q}$ by $\sqrt{p+q} - \sqrt{p-q}$.

5. Multiply $\sqrt{-b} + a$ by $\sqrt{-b} - \sqrt{-a}$.

6. $7x - 3x^2 + 14 = 0$. Complete the square and find the value of x.

7. $\sqrt{x+3} + \sqrt{3x-3} = 10$. Find the value of x.

8. In an arithmetical progression there are given the first term, 4; the number of terms, 10; and the sum of the terms, 175. Find common difference and the last term.

9. Expand by the Binomial Formula $(2a^{\frac{2}{3}} - 3b^{\frac{3}{4}})^5$.

SEPTEMBER, 1887.

1. Given $3x - \frac{4y-6}{3} = 4 - \frac{2x-4}{5}$,

$$2y - \frac{3x-2}{4} = 5\tfrac{1}{2} - \frac{3y-5}{8},$$

to find values of x and y.

2. Add $\sqrt[3]{16x^4y^3}$, $\sqrt[6]{4x^8y^6}$, and $6xy\sqrt[9]{8x^3}$.

3. A man bought a certain number of eggs for 2 dollars. If he had paid 5 cents more per dozen, he would have received two dozens less for the same money. How many dozens did he buy, and what did he pay per dozen?

4. $2\sqrt{2x-3} - \sqrt{3x-7} = \sqrt{4x-11}$. Find values of x.

5. $3x^2 - 4x = 55$. Find values of x.

6. In an arithmetical progression, given the first term, 3; the number of terms, 15; the sum of the terms, -165; to find the common difference and last term.

7. Expand $(2x - 3y^2)^5$ by the Binomial Formula.

JUNE, 1888.

(Omit one from each set.)

I.

1. Resolve $64x^7 - xy^6$ into five factors.

2. Simplify $\dfrac{\frac{x^2}{y} + \frac{y^2}{x}}{x + \frac{y^2}{x} - y} \div \left(1 + \frac{x}{y}\right)$.

3. Given $\frac{a}{x} + \frac{b}{y} = c$, $\frac{a'}{x} + \frac{b'}{y} = c'$, to find values of x and y.

II.

1. Add $3x\sqrt{a^3 - a^2 x}$, $-4a\sqrt{4ax^2 - 4x^3}$, and $5\sqrt{a^3x^2 - a^2x^3}$.

2. Multiply $2\sqrt[3]{a} - \sqrt{-x}$ by $3\sqrt{-a} + 2\sqrt[3]{x}$.

3. Given $\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}$, to find values of x.

III.

1. Given $2x^2 - 3xy + y^2 = 35$, $2x - 3y = 13$, to find values of x and y.

2. Expand $(2a - 3b)^4$ by Binomial Formula.

3. In an arithmetical progression, given the first term, 14; the number of terms, 7; the sum of the terms, $59\frac{1}{2}$; find the common difference and last term.

September, 1888.

1. Find value of

$$x(y+z) + y[x - (y+z)] - z[y - x(z - x)]$$

when $x = 3$, $y = 2$, $z = 1$.

2. A and B set out at the same time from the same spot to walk to a place 6 miles distant and back again. After walking for 2 hours, A meets B coming back. Supposing B to walk twice as fast as A, and each to maintain uniform speed throughout, find their respective rates of walking.

3. Solve the equation $\sqrt{x} + \sqrt{4+x} = \frac{4}{\sqrt{x}}$.

4. Solve the equation $\frac{5}{x+2} - \frac{2x-3}{2(x-2)} = -\frac{3}{6}$.

5. Find the sum of 10 terms of the geometrical progression in which the fourth term is 1 and the ninth term is $\frac{1}{243}$.

June, 1889.

(Omit one from each set.)

I.

1. Find the lowest common multiple of $6x^3 + 11x^2 - 46x + 24$, and $12x^3 + 37x^2 - 42x + 8$.

2. Simplify

$$\frac{x-y}{(x+z)(y+z)} + \frac{y-z}{(x+y)(x+z)} - \frac{z-x}{(x+y)(y+z)}.$$

3. Solve $$\frac{\frac{3x}{4} - \frac{y}{3}}{\frac{1}{2}} - \frac{\frac{x}{2} + \frac{2y}{5}}{\frac{13}{4}} = -\frac{7}{6}.$$

$$2y - 3x = 23.$$

II.

1. A and B run a race of 480 feet. The first heat, A gives B a start of 48 feet, and beats him by 6 seconds; the second heat, A gives B a start of 144 feet, and is beaten by 2 seconds. How many feet can each run in a second?

2. Solve $\sqrt{3x + 10} - \sqrt{3x + 25} = -3$.

3. Solve $\dfrac{2x^2 + 3x - 5}{3x^2 + 4x - 1} = \dfrac{2x^2 - x - 1}{3x^2 - 2x + 7}$.

III.

1. In an arithmetical progression, given the first term, -3; the common difference, $2\frac{1}{3}$; and the sum of the terms, 143; to find the last term and the number of terms.

2. In a geometrical progression, prove the formula for the sum of n terms.

3. Solve $2x^2 - 3y^2 = 60$, and $3x^2 - 4xy + y^2 = 64$.

EXAMINATION PAPERS FOR ADMISSION TO MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

SEPTEMBER, 1884.

1. Simplify $\dfrac{\dfrac{2y+4x^2}{y^2}}{1+\dfrac{4x^2}{y^2}}$, after substituting $1-x^2$ for y.

2. Resolve $a^{12}-b^{12}$ into six factors.

Solve the following equations:

3. $\dfrac{x}{a}+\dfrac{y}{b}=1,\quad \dfrac{x}{b}-\dfrac{y}{a}=1.$

4. $\dfrac{x^2+2x}{2}-\dfrac{x^2-x+2}{4}+\dfrac{x-2}{3}=0.$

5. $\sqrt{2x+3}-\sqrt{x+1}=\sqrt{3x-8}.$

6. Prove that the square of half the sum of any two unequal numbers is less than half the sum of their squares.

7. Expand by the Binomial Theorem $\left(a-\dfrac{2b^2}{a}\right)^4$.

8. Insert two arithmetical means between 24 and 81. Also insert two geometrical means between the same numbers.

JUNE, 1885.

1. Divide a^3-a^2 by $a^{\frac{1}{2}}-a^{\frac{1}{3}}$.

2. Factor $x^2-x-30,\quad (x-y)^3-y^3,\quad x^{4n+1}-x.$

3. Find the value of $\dfrac{x^2-y^2}{x^2+y^2}$, when $x=\dfrac{a+b}{a-b}$ and $y=\dfrac{a-b}{a+b}$.

Solve the following equations:

4. $\dfrac{x-a}{x-b} + 2\,\dfrac{x-b}{x-c} = 3.$

5. $\dfrac{x + 3\,x^{-1}}{1 + 2\,x^{-1}} = 3 - x.$

6. $x - \sqrt[3]{x^3 - 2\,x^2} = 2.$

7. $(x+2)\,(y-3) = 10, \qquad x\,y = 15.$

8. Find the cube of $1 + \sqrt{-3}$.

9. The sum of three terms in arithmetical progression beginning with $\frac{3}{2}$ is equal to the sum of three terms in geometrical progression beginning with $\frac{3}{2}$, and the common difference is equal to the ratio. What are the two series?

September, 1885.

1. Factor $x^2 - 6\,x - 16$ and $1 - 9\,a + 8\,a^2$.

2. Find highest common factor of $x^2 + x - 6$ and $2\,x^2 - 11\,x + 14$.

3. Simplify $\dfrac{x}{x-y} + \dfrac{3\,x}{x+y} - \dfrac{2\,x\,y}{x^2 - y^2}$.

Solve the following equations:

4. $\dfrac{x}{2} + \dfrac{1 - 2\,a\,x}{2\,a} + \dfrac{2\,x - 1}{a^2} = 0.$

5. $\dfrac{7}{x^2 - 4} - \dfrac{3}{x+2} = \dfrac{22}{5}.$

6. $\sqrt{x - 32} + \sqrt{x} = 16.$

7. Find the sum of 16 terms of the arithmetical progression $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, . . .

8. Find the sum to infinity of the geometrical series $1, -\frac{a^2}{x^2}, \frac{a^4}{x^4}, \ldots$

9. Expand $(3x - 2y)^5$ by the Binomial Theorem.

JUNE, 1886.

1. Find the value of $\frac{x-a}{b} - \frac{x-b}{a}$, when $x = \frac{a^2}{a-b}$.

2. Add together $\frac{x-a}{(x+a)^2}$, $\frac{x+a}{(x-a)^2}$, $\frac{2x(3a-x)}{(x-a)(x+a)^2}$.

3. Solve $\frac{1}{2x-1} + \frac{2}{4x-3} = \frac{1}{x-1}$.

4. Solve $x + \sqrt{x^2 - a^2} = b$.

5. Show that $\frac{x^n - \sqrt{x^{2n} - 4}}{2}$ is the reciprocal of $\left(\frac{\sqrt{x^n + 2} + \sqrt{x^n - 2}}{2}\right)^2$.

6. Show that $(-1 + \sqrt{-3})^3 + (-1 - \sqrt{-3})^3 = 16$.

7. Solve $\frac{6}{x} + \frac{x}{6} = \frac{5(x-1)}{4}$.

8. Solve $x^2 + xy = 15$, $xy - y^2 = 2$.

9. Find the 4th term of $(a - 2b)^{10}$.

10. How many terms of $16 + 24 + 32 + 40 + \ldots$ amount to 1840?

SEPTEMBER, 1886.

1. Simplify $\frac{x+1}{x^2-2x}+\frac{2x-1}{x^2-x-2}-\frac{3x+2}{x^2+x}$.

2. Resolve $a^{12}-b^{12}$ into its prime factors.

3. Solve $\frac{x-5}{6}+\frac{5x-7}{9}-\frac{3x-7}{4}=\frac{5-x}{3}$.

4. Find the continued product of

$$\sqrt{a+b},\ \sqrt[4]{a-b},\ \text{and}\ \sqrt[4]{(a^2-b^2)^3}.$$

5. Extract the square root of $41+12\sqrt{5}$.

Solve the following equations:

6. $\frac{a}{x}+\frac{x}{b}+\frac{b}{a}=0$.

7. $\sqrt{x+\frac{1}{3}}=\sqrt[3]{x+\frac{1}{6}}$.

8. $\begin{cases}\sqrt{2x-y}=\sqrt{x-y}+1,\\ x^2+4y=17.\end{cases}$

9. There are two numbers whose geometrical mean is $\frac{4}{5}$ of their arithmetical mean; and if the two numbers be taken for the first two terms of an arithmetical progression, the sum of its first three terms is 36. What are the numbers?

JUNE, 1887.

1. Reduce to its lowest terms $\frac{x^4+3x-2}{x^4+3x^2+4}$.

2. Solve $\frac{a}{b-x}=\frac{b}{a-x}$.

3. Simplify $\left[\frac{(a^{m-n})^{m+n}(a^n)^{n+p}}{(a^m)^{m-n}}\right]^{\frac{1}{n}}$.

4. Solve $\sqrt{2x+1} - \sqrt{x+3} = \sqrt{x}$.

5. Solve $x^2 + y^2 = 20$, $x^2 - xy = 8$.

6. Solve $2x^3 + 8x^{-3} = 17$.

7. Reduce $\dfrac{5 + 3\sqrt{-1}}{1 + \sqrt{-1}}$ to the form $A + B\sqrt{-1}$.

8. The 1st term of an arithmetical progression is 2, and the difference between the 3d and 7th terms is 6. Find the sum of the first 12 terms.

SEPTEMBER, 1887.

1. Divide $x^2 + x^{-2}$ by $x^{\frac{2}{3}} + x^{-\frac{2}{3}}$.

2. Resolve into two factors $a^2 + b^2 - c^2 - d^2 + 2(ab + cd)$.

3. Solve $x - a + \sqrt{x^2 - 2ax} = b$.

4. Solve $\dfrac{3x - \sqrt{x^2 - 8}}{x - \sqrt{x^2 - 8}} = x + \sqrt{x^2 - 8}$.

5. Solve $(x - y)(x - 3y) = 24$, $x - 2y = 5$.

6. Form the quadratic equation whose roots are $a + b - c$ and $a - b + c$.

7. Insert three geometrical means between $3\frac{5}{9}$ and 18.

8. Give the first, third, and fifth terms of the expansion by the Binomial Theorem of $\left(x\sqrt{y} + \dfrac{y^2}{2\sqrt{x}}\right)^{10}$.

JUNE, 1888.

PRELIMINARY.

1. Factor $8cx - 12cy + 2ax - 3ay$,
and $2am - b^2 + m^2 + 2bn + a^2 - n^2$.

2. Find the G. C. D. & L. C. M. of $2x^4 - 11x^3 + 3x^2 + 10x$ and $3x^4 - 14x^3 - 6x^2 + 5x$.

3. Simplify $\dfrac{\dfrac{x+2y}{x+y} + \dfrac{x}{y}}{\dfrac{x+2y}{y} - \dfrac{x}{x+y}}$ and $\dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{x}}}$.

4. Solve the equations

(a.) $\dfrac{5 - 2x}{x + 1} = \dfrac{3 - 2x}{x + 4}$,

(b.) $\dfrac{x}{2a} - 3 + \dfrac{x}{4a^3} = \dfrac{x}{3a^2} - 2a(2 - 3a)$.

5. At what time between 4 and 5 o'clock is the minute hand of a watch exactly 5 minutes in advance of the hour hand?

6. Solve the simultaneous equations

$$\begin{aligned} 5x - 3y + 2z &= 41. \\ 2x + y - z &= 17. \\ 5x + 4y - 2z &= 36. \end{aligned}$$

7. Extract the square root of

$$x^2 + 4y^2 + 9z^2 - 4xy + 6xz - 12yz.$$

8. Reduce to an equivalent fraction having a rational denominator

$$\frac{\sqrt{x} - 4\sqrt{x-2}}{2\sqrt{x} + 3\sqrt{x-2}}.$$

FINAL.

1. Solve the equation $2x^2+3x-5\sqrt{2x^2+3x+9}=-3$.

2. Solve the simultaneous equations

$$\begin{cases} x^2+xy+4y^2=6. \\ 3x^2+8y^2=14. \end{cases}$$

3. Factor $x^4-7x^2y^2+y^4$.

4. A person saves \$270 the first year, \$210 the second, and so on. In how many years will a person who saves every year \$180 have saved as much as he?

5. Expand $\left(m^{-\frac{3}{5}}+2n^3\right)^7$.

Find 5th term of $\left(x^{-1}-2y^{\frac{1}{2}}\right)^{11}$.

6. Form the equation whose roots are $-\frac{8}{3}$ and $\frac{4}{7}$.

7. Derive the formula for the sum of a series in geometrical progression.

8. Find three numbers in geometrical progression such that their sum shall be 14 and the sum of their squares 84.

COMPLETE.

1. Simplify $\dfrac{a^3-b^3}{a^4-b^4}-\dfrac{a-b}{a^2-b^2}-\dfrac{1}{2}\left(\dfrac{a+b}{a^2+b^2}-\dfrac{1}{a+b}\right)$.

2. Solve $\dfrac{x}{a+b}+\dfrac{y}{a-b}=2,\ x+y=2a$.

3. Extract the square root of

$$4a^4-12a^3b+29a^2b^2-30ab^3+25b^4$$

4. Simplify $\left(\dfrac{5+2\sqrt{3}}{4-\sqrt{3}}\right)^2\times\left(\dfrac{2-\sqrt{3}}{\sqrt{3}+1}\right)^2$.

5. Solve $\dfrac{9x-1}{x-\frac{1}{x}} = \dfrac{55}{6}$.

6. Solve $\sqrt{a+x} + \sqrt{a-x} = 2\sqrt{x}$.

7. Solve $x - \dfrac{x-y}{2} = 4,\quad y - \dfrac{x+3y}{x+2} = 1$.

8. Find the sum of 18 terms of the series $\frac{2}{3}, -1, -2\frac{2}{3}, \ldots$

SEPTEMBER, 1888.

1. Reduce to its lowest terms $\dfrac{4x^2+3x-10}{4x^3+7x^2-3x-15}$.

2. Simplify $\left(\sqrt{\dfrac{a+x}{x}} - \sqrt{\dfrac{x}{a+x}}\right)^2 - \left(\sqrt{\dfrac{x}{a}} - \sqrt{\dfrac{a}{x}}\right)^2$.

3. A fraction becomes $\frac{3}{4}$ by the addition of 3 to the numerator and 1 to the denominator. If 1 be subtracted from the numerator and 3 from the denominator it becomes $\frac{1}{2}$. Find the fraction.

4. Solve $(x-a)^2 = (x-2a)(x^2+4a^2)^{\frac{1}{2}}$.

5. Form the quadratic equation whose roots are

$$\frac{(a+b)^2}{a-b} \text{ and } b-a.$$

6. Expand by the Binomial Theorem $\left(x + \dfrac{1}{x}\right)^7$.

7. Divide 111 into three parts such that the products of each pair may be in the ratios 4 : 5 : 6.

8. Find the sum to infinity of the geometrical progression $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \ldots$

MAY, 1889.

PRELIMINARY.

1. Find the greatest common divisor of $2x^4 - 12x^3 + 19x^2 - 6x + 9$ and $4x^3 - 18x^2 + 19x - 3$.

2. Simplify $\dfrac{a^3 - b^3}{a^3 + b^3} \times \dfrac{a + b}{a - b} \times \dfrac{(a^2 - ab + b^2)^2}{(a^2 + ab + b^2)^2}$.

3. One tap will empty a vessel in 80 minutes, a second in 200 minutes, and a third in 5 hours. How long will it take to empty the vessel if all the taps are opened?

4. Solve $\dfrac{x}{a+b} + \dfrac{y}{a-b} = 2a$, $\dfrac{x-y}{4ab} = 1$.

5 Extract the square root of $x^4 - x^3 + \dfrac{x^2}{4} + 4x - 2 + \dfrac{4}{x^2}$.

6. Factor $2am - b^2 + m^2 + 2bn + a^2 - n^2$, and $2c^3m + 8c^2m - 42cm$.

7. Which is the greater, $\sqrt{10}$ or $\sqrt[5]{46}$, and why?

8. Extract the square root of $75 - 12\sqrt{21}$.

FINAL.

1. Write out the first four terms, the last four terms, and the middle term of $(x - 2y)^{14}$.

2. Find the sum of the first n terms of the series 1, 2, 3, . . .

3. Find three geometrical means between 2 and 162.

4. Show that in the equation $x^2 + px + q = 0$, the sum of the roots is $-p$, and the product of the roots q.

5. Find the four roots of the equation $x^4 - 3x^2a^2 + a^4 = 0$.

6. A number consists of two figures whose product is 21; and if 22 is subtracted from the number and the sum of the squares of its figures added to the remainder, the order of the figures will be inverted. What is the number?

7. Solve $3x^2 + 15x - 2\sqrt{x^2 + 5x + 1} = 2$.

8. Form the equation whose roots are $(a - \frac{1}{2})$, $(b + \frac{2}{3})$.

COMPLETE.

1. Simplify

$$\left(x - \frac{xy - y^2}{x + y}\right)\left(x - \frac{xy^2 - y^3}{x^2 + y^2}\right) \div \left(1 - \frac{xy - y^2}{x^2}\right).$$

2. Reduce to its lowest terms $\dfrac{x^4 - 2x^3 + 2x - 1}{x^6 - 15x^2 + 24x - 10}$.

3. Simplify $\dfrac{(a + b)^{\frac{1}{2}} + (a - b)^{\frac{1}{2}}}{(a + b)^{\frac{1}{2}} - (a - b)^{\frac{1}{2}}} + \dfrac{(a + b)^{\frac{1}{2}} - (a - b)^{\frac{1}{2}}}{(a + b)^{\frac{1}{2}} + (a - b)^{\frac{1}{2}}}$.

4. A certain number when divided by a second gives a quotient 3 and a remainder 2; if 9 times the second number be divided by the first, the quotient is 2 and the remainder 11. Find the two numbers.

5. Solve $x^{\frac{1}{2}} - a^{\frac{1}{2}} = (x - b)^{\frac{1}{2}}$.

6. Solve $x^4 + 4abx^2 = (a^2 - b^2)^2$.

7. If A is the sum of the odd terms, and B of the even terms, in the expansion of $(x + a)^4$, show that $A^2 - B^2 = (x^2 - a^2)^4$.

8. If $x - y$ is a mean proportional between y and $y + z - 2x$, show that x is a mean proportional between y and z.

9. The second term of a geometrical progression is 54, and the fifth term 16. Find the series.